On quasivarieties of graphs

Marina Schwidefsky[1] Anna Zamojska-Dzienio[2]

[1]Institute of Mathematics of the Siberian Branch of Russian Academy of Sciences
[2]Faculty of Mathematics and Information Science, Warsaw University of Technology

AAA92, Prague

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A **quasi-equational basis** of $\mathbf{K}$ is any defining set of quasi-identities for $\mathbf{Q}(\mathbf{K})$.

A **(directed) graph** $\mathcal{G} = (G, E)$ is a set G endowed with a binary relation $E \subseteq G^2$. A graph $\mathcal{G}$ is **antireflexive** if:

$$\mathcal{G} \models \forall xy \; E(x,x) \longrightarrow x = y.$$

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The Birkhoff-Malcev Problem: Which lattices are isomorphic to lattices of subquasivarieties?

Q-universality

A quasivariety **K** of a finite type is Q-**universal** if for any quasivariety **M** of a finite type, Lq(**M**) is a homomorphic image of a sublattice of Lq(**K**). (M.V. Sapir, 1985)

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Quasivariety lattices of Q-universal quasivarieties are *complex*.

Let **C** be the quasivariety of antireflexive graphs defined by the following quasi-identities:

$$\forall\, xyz\ E(x,z)\ \&\ E(y,z) \ \longrightarrow\ x=y;$$
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Theorem (A. Kravchenko, 1997)

The quasivariety **C** *is Q-universal.*

Directed cycles

$\mathcal{C}_1 = \langle \{0\}; \{(0,0)\} \rangle$ - the trivial graph.

For an integer $n > 1$,

$$\mathcal{C}_n = \langle \{0, \ldots, n-1\}; E \rangle$$

denote the graph such that for any $i, j < n$,

$$(i,j) \in E \quad \text{if and only if} \quad j \equiv i + 1 (\mathrm{mod}\, n).$$

The graph $\mathcal{C}_n$ is called the **directed cycle of length** n.

$\mathcal{C}_n \in \mathbf{C}$ for any $n > 0$.

Unreasonability

Theorem (A. Nurakunov, 2012)

Let a σ *contain a non-constant non-idempotent operation. Then there is a quasivariety* $\mathbf{K}$ *of signature* σ *such that the set of all finite sublattices of* $\mathrm{Lq}(\mathbf{K})$ *is not computable.*

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Remark. It means that there is no algorithm to decide whether a given finite lattice embeds into such a quasivariety lattice.

Unreasonability for graphs

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There is a class $\mathbf{K} \subseteq \mathbf{C}$ *such that the set of isomorphism types of the class of finite sublattices of* $\mathrm{Lq}(\mathbf{K})$ *is computably enumerable but not computable.*

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There are countably many of such classes in the first case, and continuum many in the second one.

Corollaries

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There are continuum many quasivarieties $\mathbf{R}$ *of graphs such that the finite membership problem for* $\mathbf{R}$ *and the quasi-equational theory of* $\mathbf{R}$ *are undecidable.*

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A set Σ of quasi-identities such that $\mathbf{K} = \mathrm{Mod}(\Sigma)$ is called a **basis** of $\mathbf{K}$.

There are continuum many quasivarieties of graphs which do not have a computable basis of quasi-identities.

Independent basis

A basis of **K** is **independent** if none of its proper subsets is a basis of **K**.

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There are uncountably many quasivarieties of graphs which have no independent basis of quasi-identities.

Differential groupoids

Similar results can be obtained for **differential groupoids**:

$$
\begin{aligned}
x \cdot x &= x \\
(x \cdot y) \cdot (z \cdot t) &= (x \cdot z) \cdot (y \cdot t) \\
x \cdot (x \cdot y) &= x.
\end{aligned}
$$

THANK YOU FOR YOUR ATTENTION.