On the poset of minors of a function

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Functions

Definition

Function of n variables:

$$f\colon A^n \to B.$$

The set of all such functions is denoted by $\mathcal{F}_{AB}$.

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  (game theory, operations research)
- $A = B$: operations
  (algebra)

Minors

Definition

The function g is a minor of f (notation: $g \leq f$) if g can be obtained from f by substituting variables to variables:

$$g \leq f \iff \exists i_1, \ldots, i_n \in \{1, \ldots, k\} : \; g\left(x_1, \ldots, x_k\right) = f\left(x_{i_1}, \ldots, x_{i_n}\right).$$

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Examples

- $g(x, y, z) = f(y, x, z)$ permutation of variables
- $g(x, y, z) = f(x, y)$ introduction of inessential variables
- $g(x, y) = f(x, y, y)$ identification of variables

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- $g(x, y, z) = f(x, y)$ — introduction of inessential variables
- $g(x, y) = f(x, y, y)$ — identification of variables

Our main object of study is the poset $(\mathcal{F}_{AB} / \equiv; \leq)$.

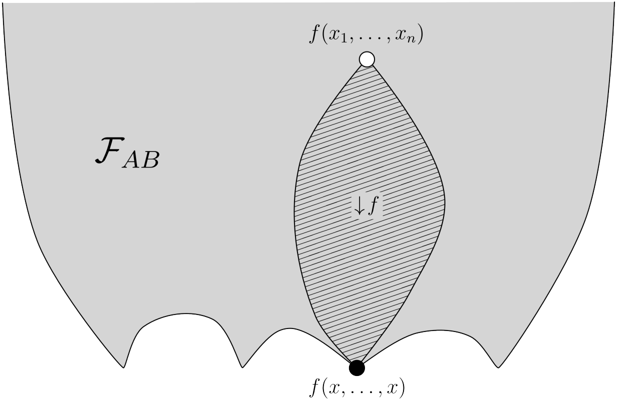
Minor posets

Minor posets

Definition

A poset P is a minor poset if there is a function $f \in \mathcal{F}_{AB}$ (for some sets A and B) such that P is isomorphic to the poset of minors of f, i.e.,

$$P \cong {\downarrow} f := (\{g \in \mathcal{F}_{AB} \mid g \leq f\} / \equiv; \leq) .$$



Partitions and minors

When forming a minor of $f(x_1, \ldots, x_n)$, it is sufficient to tell which variables are being identified with each other. This can be given by a partition of $\{1, \ldots, n\}$.

The set of all partitions of $\{1, \ldots, n\}$ forms the partition lattice Π_n.
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Example

Let f be the Boolean function given by the following polynomial over $\mathbb{Z}_2$:

$$f(x_1, x_2, x_3, x_4) = x_1 x_3 + x_2 + x_4.$$

For $\alpha = 1 \mid 24 \mid 3$ we obtain the minor

$$f_\alpha(x, y, z) = f(x, y, z, y) = xz + y + y = xz.$$

Partitions and minors

Let $f(x_1, x_2, x_3, x_4) = x_1 x_3 + x_2 + x_4$.

- $\alpha = 1 \mid 2 \mid 3 \mid 4 \quad f_\alpha(x, y, z, u) = f(x, y, z, u) = xz + y + u$
- $\alpha = 12 \mid 3 \mid 4 \quad f_\alpha(x, y, z) = f(x, x, y, z) = xy + x + z$
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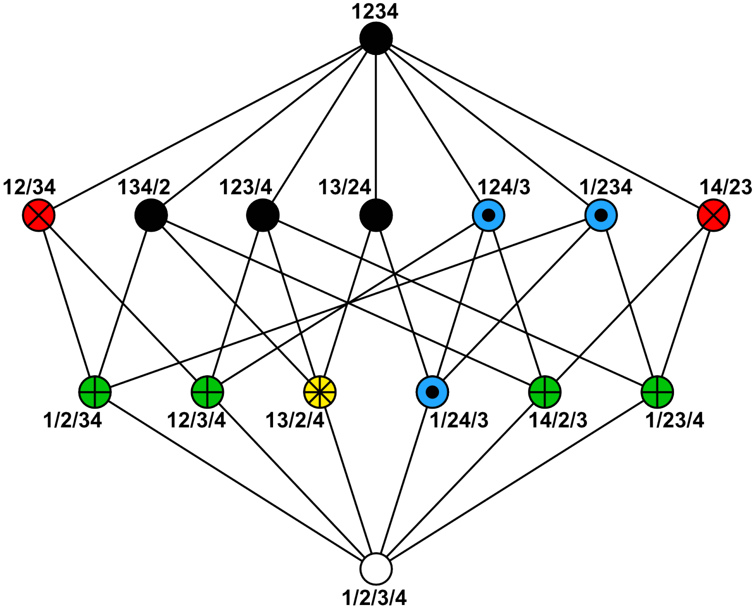
Colored partition lattice



Minor poset from colored partition lattice

Let us color the partition lattice by the minors of f (up to equivalence):

$$c \colon \Pi_n \to {\downarrow} f, \quad \alpha \mapsto f_\alpha / {\equiv} \, .$$

The resulting quotient poset is isomorphic to the dual of the minor poset of f:

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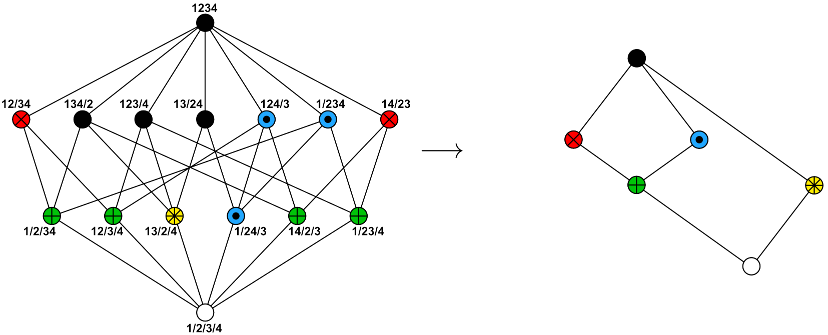

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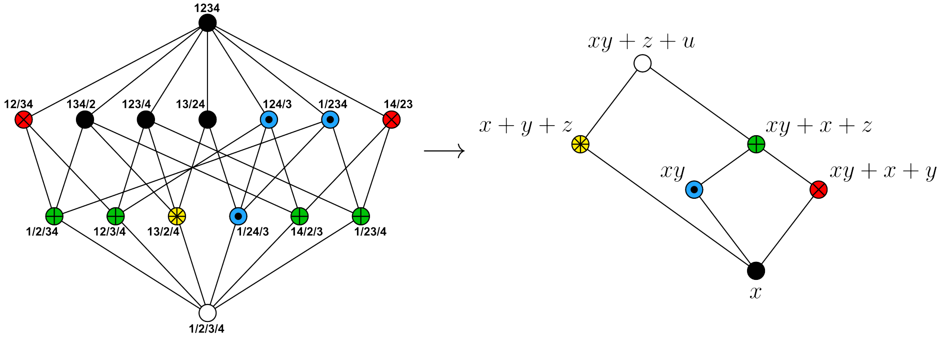

The main problem

Problem

Which finite bounded posets are minor posets?

Problem solved!

Since the minor poset of f is determined by the "minor coloring", it suffices to describe these colorings.

Theorem

For every coloring $c\colon \Pi_n \to C$, the following two conditions are equivalent.

(i) *There is a function $f\colon A^n \to B$ such that*

$$\forall \alpha, \beta \in \Pi_n\colon \; c(\alpha) = c(\beta) \iff f_\alpha \equiv f_\beta.$$

(ii) *. . . (Very Technical Condition) . . .*

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$$\forall \alpha, \beta \in \Pi_n\colon\ c(\alpha) = c(\beta) \iff f_\alpha \equiv f_\beta.$$

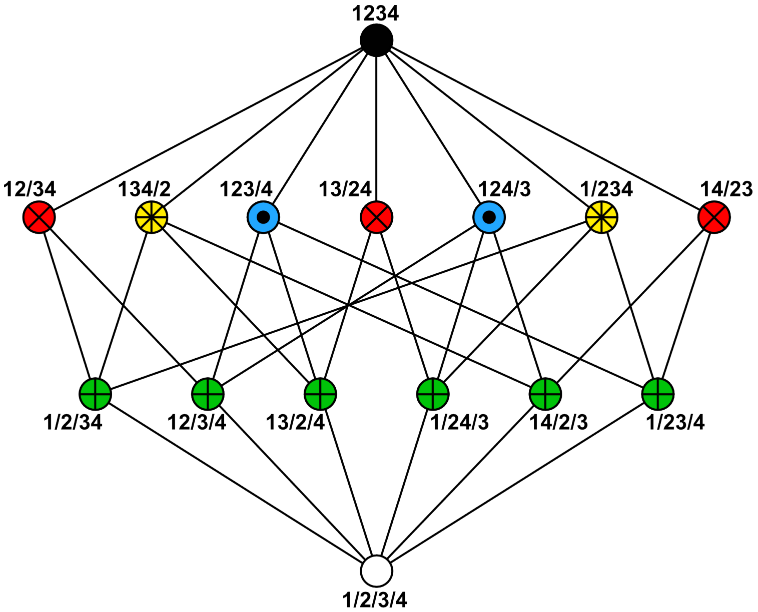
(ii) *For all $\alpha, \beta \in \Pi_n$, if $c(\alpha) = c(\beta)$, then there exist $\alpha_0, \ldots, \alpha_k, \beta_0, \ldots, \beta_\ell \in \Pi_n$ (for some $k, \ell \in \mathbb{N}_0$), such that $\alpha_0 = \alpha$, $\beta_0 = \beta$, and*

(a) $[\alpha_k; \top]$ *and* $[\beta_\ell; \top]$ *are isomorphic as colored posets;*

(b) $\forall i \in \{0, \ldots, k-1\}\colon\ \alpha_i \prec \alpha_{i+1}$ *and* $\exists \eta_i \in \Pi_n$ *s.t.* $\alpha_i \leq \eta_i \prec \top,\ \alpha_{i+1} \not\leq \eta_i$, *and* $\forall \gamma \in \Pi_n\colon\ \alpha_i \prec \gamma \not\leq \eta_i \implies \forall \xi \in [\alpha_i; \eta_i] : c(\xi) = c(\xi \vee \gamma)$;

(c) $\forall j \in \{0, \ldots, \ell-1\}\colon\ \beta_j \prec \beta_{j+1}$ *and* $\exists \vartheta_j \in \Pi_n$ *s.t.* $\beta_j \leq \vartheta_j \prec \top,\ \beta_{j+1} \not\leq \vartheta_j$, *and* $\forall \gamma \in \Pi_n\colon\ \beta_j \prec \gamma \not\leq \vartheta_j \implies \forall \xi \in [\beta_j; \vartheta_j] : c(\xi) = c(\xi \vee \gamma)$.

Instead of VTC: What is wrong with this coloring?



Problem solved?

Despite the characterization of "minor colorings", it is still not easy tell whether a given poset is a minor poset or not. Some partial results:

Theorem

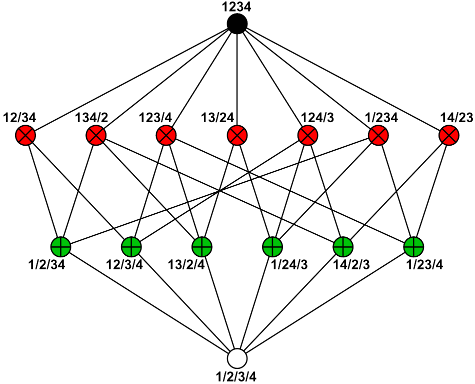
All bounded posets with at most 6 elements are minor posets.

Theorem

For every $n \in \mathbb{N}$, the following posets are minor posets:

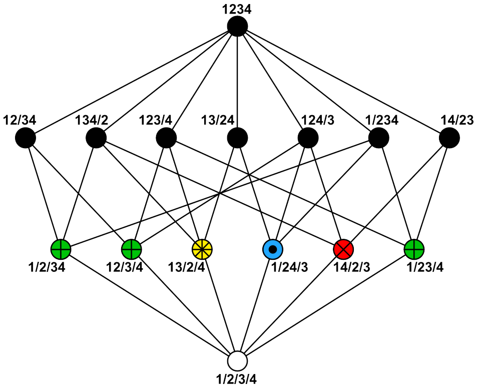
- *the chain of length n;*
- *the lattice M_n;*
- *the n-dimensional cube $\mathbf{2}^n$.*

Proof: the chain



$\longrightarrow$

Proof: M_n



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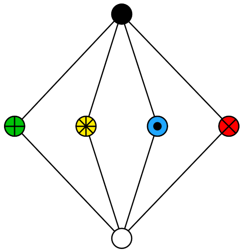
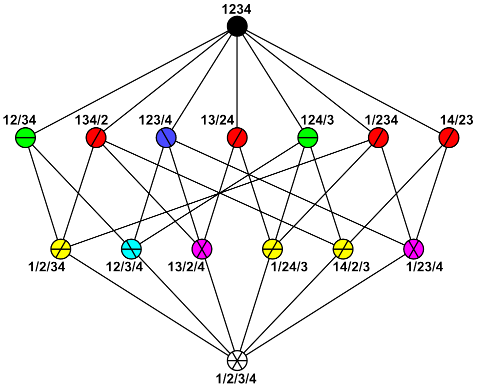


Proof: the cube



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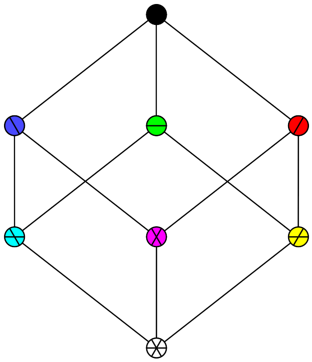
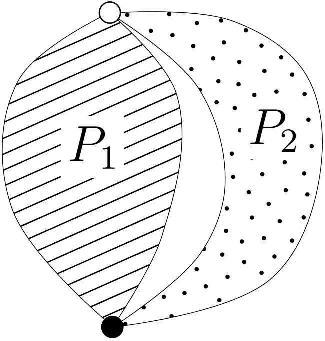
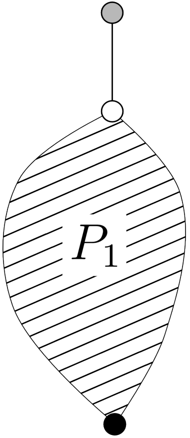


Constructions

Theorem

If P_1 and P_2 are minor posets, then so are the following:





Last slide, last question

After all, is there a finite bounded poset that is not a minor poset?