On representation of lattice-valued frames as quantale algebras

Sergejs Solovjovs

Institute of Mathematics, Faculty of Mechanical Engineering
Brno University of Technology
Technicka 2896/2, 616 69, Brno, Czech Republic
e-mail: solovjovs@fme.vutbr.cz

92nd Workshop on General Algebra

Czech University of Life Sciences, Prague, Czech Republic
May 27 – 29, 2016

Acknowledgements

The author gratefully acknowledges the support of the bilateral project "New Perspectives on Residuated Posets" of the Austrian Science Fund (FWF) (project No. I 1923-N25) and the Czech Science Foundation (GAČR) (project No. 15-34697L).

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Outline

Origins of lattice-valued frames

- Equivalence of the categories of sober topological spaces (point-set topology) and spatial locales (point-free topology) disclosed a relationship between general topology and universal algebra.
- The concept of fuzzy topological space inspired researchers to provide a fuzzy analogue of the sobriety-spatiality equivalence.
- Lattice-valued point-free topology needed a lattice-valued analogue of locales different from that of lattice-valued algebra.

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Existing concepts of lattice-valued frame

D. Zhang, Y.-M. Liu (1995): L-fuzzy frames are objects of the comma category $(L \downarrow \mathbf{Frm})$, i.e., frame homomorphisms $L \xrightarrow{i_A} A$.

A. Pultr, S. E. Rodabaugh (2003): lattice-valued frames are families of frame homomorphisms $(A_1 \xrightarrow{\varphi_t} A_2)_{t \in T}$ with some properties.

W. Yao (2011): L-frames are based in the notion of L-partially ordered set; L-frames and L-fuzzy frames are categorically equivalent.

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Quantale algebras as lattice-valued frames

- Quantale algebras are motivated by algebras over a not necessarily commutative unital ring.
- Quantale algebras (crisp concept) incorporate L-fuzzy frames of D. Zhang, Y.-M. Liu and L-frames of W. Yao (fuzzy concept).
- Quantale algebras provide a framework for developing fuzzy analogues of the sobriety-spatiality equivalence.
- Quantale algebras (crisp concept) facilitate the study of the categories of different lattice-valued structures (fuzzy concept).

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Quantales

Definition 1

- A quantale is a triple $(Q, \bigvee, \otimes)$ such that
  - $(Q, \bigvee)$ is a $\bigvee$-semilattice;
  - $(Q, \otimes)$ is a semigroup;
  - $q \otimes (\bigvee S) = \bigvee_{s \in S}(q \otimes s)$ and $(\bigvee S) \otimes q = \bigvee_{s \in S}(s \otimes q)$ for every $q \in Q$, $S \subseteq Q$.
- A quantale homomorphism $(P, \bigvee, \otimes) \xrightarrow{\varphi} (Q, \bigvee, \otimes)$ is a map $P \xrightarrow{\varphi} Q$ which preserves $\bigvee$ and $\otimes$.
- **Quant** is the category of quantales.

Unital quantales

Definition 2

- A quantale Q is said to be unital provided that there exists an element $1 \in Q$ such that $(Q, \otimes, 1)$ is a monoid.
- Unital quantale homomorphisms additionally preserve the unit.
- **UQuant** is the subcategory of **Quant** of unital quantales.

Quantale modules

Definition 3

- Given a unital quantale Q, a (unital left) Q-module is a pair $(A, *)$, where A is a $\bigvee$-semilattice and $Q \times A \xrightarrow{*} A$ is a map (the action of Q on A) such that
  - $q * (\bigvee S) = \bigvee_{s \in S}(q * s)$ for every $q \in Q$, $S \subseteq A$;
  - $(\bigvee S) * a = \bigvee_{s \in S}(s * a)$ for every $S \subseteq Q$, $a \in A$;
  - $q_1 * (q_2 * a) = (q_1 \otimes q_2) * a$ for every $q_1, q_2 \in Q$, $a \in A$;
  - $1_Q * a = a$ for every $a \in A$.
- A Q-module homomorphism $(A, *) \xrightarrow{\varphi} (B, *)$ is a $\bigvee$-preserving map $A \xrightarrow{\varphi} B$ with $\varphi(q * a) = q * \varphi(a)$ for every $q \in Q$, $a \in A$.
- Q-**Mod** is the category of Q-modules.

Quantale algebras

Definition 4

- For a unital quantale Q, a Q-algebra is a triple $(A, \otimes, *)$ where
  - $(A, *)$ is a Q-module;
  - $(A, \otimes)$ is a quantale;
  - $q * (a_1 \otimes a_2) = (q * a_1) \otimes a_2 = a_1 \otimes (q * a_2)$ for every $q \in Q$, $a_1, a_2 \in A$.
- A Q-algebra homomorphism $(A, \otimes, *) \xrightarrow{\varphi} (B, \otimes, *)$ is a map $A \xrightarrow{\varphi} B$ which is a homomorphism of quantales and Q-modules.
- Q-**Alg** is the category of Q-algebras.

Preliminary definitions

Definition 5

Given a unital quantale Q, Q-**UAlg** is the subcategory of Q-**Alg** of unital quantale algebras and unit-preserving homomorphisms.

Definition 6

Given a unital quantale Q, $(Q \downarrow \mathbf{UQuant})_z$ is the category, whose

objects are **UQuant**-morphisms $Q \xrightarrow{i_A} A$ having their range in the center $Z(A) := \{a \in A \mid a \otimes a' = a' \otimes a \text{ for every } a' \in A\}$ of A;

morphisms $(Q \xrightarrow{i_A} A) \xrightarrow{\varphi} (Q \xrightarrow{i_B} B)$ are **UQuant**-morphisms $A \xrightarrow{\varphi} B$ such that $\varphi \circ i_A = i_B$.

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Quantale algebras as comma categories

Theorem 7

Let Q be a unital quantale.

1. *There is a functor* $Q\text{-}\mathbf{UAlg} \xrightarrow{F} (Q \downarrow \mathbf{UQuant})_z$, $F(A,*) = (Q \xrightarrow{i_A} A)$, $F\varphi = \varphi$, *where* $i_A(q) = q * 1_A$.
2. *There is a functor* $(Q \downarrow \mathbf{UQuant})_z \xrightarrow{G} Q\text{-}\mathbf{UAlg}$, $G(Q \xrightarrow{i_A} A) = (A,*)$, $G\varphi = \varphi$, *where* $q * a = i_A(q) \otimes a$.
3. $G \circ F = 1_{Q\text{-}\mathbf{UAlg}}$ *and* $F \circ G = (Q \downarrow \mathbf{UQuant})_z$, *i.e., the categories* $Q\text{-}\mathbf{UAlg}$ *and* $(Q \downarrow \mathbf{UQuant})_z$ *are isomorphic.*

Preliminary definitions

Definition 8

- Every quantale Q has a binary operation (i.e., residuation) $q_1 \rightarrow_l q_2 = \bigvee\{q \in Q \mid q \otimes q_1 \leqslant q_2\}$.
- Every Q-module $(A, *)$ has a binary operation (i.e., residuation) $a_1 \twoheadrightarrow a_2 = \bigvee\{q \in Q \mid q * a_1 \leqslant a_2\}$.

Definition 9

Given a map $f : X \to Y$ and a $\bigvee$-semilattice L, there is the forward L-powerset operator $f_L^{\rightarrow} : L^X \to L^Y$, $(f_L^{\rightarrow}(\alpha))(y) = \bigvee\{\alpha(x) \mid f(x) = y\}$.

Definition 10

Given a $\bigvee$-semilattice L and a set X, every $S \subseteq X$, $a \in L$ have a map $\alpha_S^a : X \to L$ with $\alpha_S^a(x) = a$ for $x \in S$; otherwise, $\alpha_S^a(x) = \bot$.

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Lattice-valued $\bigvee$-semilattices

Definition 11

- Given a unital quantale Q, a Q-$\bigvee$-semilattice is a triple $(A, e, \bigsqcup)$, where $A \times A \xrightarrow{e} Q$ is a map (Q-partial order) with
  - $1_Q \leqslant e(a, a)$ for every $a \in A$;
  - $e(a_2, a_3) \otimes e(a_1, a_2) \leqslant e(a_1, a_3)$ for every $a_1, a_2, a_3 \in A$;
  - $1_Q \leqslant e(a_1, a_2)$, $1_Q \leqslant e(a_2, a_1)$ imply $a_1 = a_2$, for every $a_1, a_2 \in A$;

  and $Q^A \xrightarrow{\bigsqcup} A$ is another map (Q-join operation) with $e(\bigsqcup \alpha, a) = \bigwedge_{a' \in A}(\alpha(a') \to_l e(a', a))$ for every $\alpha \in Q^A$, $a \in A$.
- A Q-$\bigvee$-semilattice homomorphism $(A, e, \bigsqcup) \xrightarrow{\varphi} (B, e, \bigsqcup)$ is a map $A \xrightarrow{\varphi} B$ such that $\varphi(\bigsqcup \alpha) = \bigsqcup \varphi_Q^{\rightarrow}(\alpha)$ for every $\alpha \in Q^A$.
- **Sup**(Q) is the category of Q-$\bigvee$-semilattices.

Example of lattice-valued $\bigvee$-semilattices

Lemma 12

Every unital quantale Q provides a Q-$\bigvee$-semilattice $(Q, e, \bigsqcup)$, where the maps e and $\bigsqcup$ are defined by

- $e(q_1, q_2) = q_1 \rightarrow_l q_2$ *for every* $q_1, q_2 \in Q$;
- $\bigsqcup \alpha = \bigvee_{q \in Q} (\alpha(q) \otimes q)$ *for every* $\alpha \in Q^Q$.

Quantale modules as lattice-valued $\bigvee$-semilattices

Theorem 13

Let Q be a unital quantale.

1. *There exists a functor* $Q\text{-}\mathbf{Mod} \xrightarrow{F} \mathbf{Sup}(Q)$, $F(A,*) = (A, e, \bigsqcup)$, $F\varphi = \varphi$, *where*
   - $e(a_1, a_2) = a_1 \twoheadrightarrow a_2$;
   - $\bigsqcup \alpha = \bigvee_{a \in A}(\alpha(a) * a)$.
2. *There exists a functor* $\mathbf{Sup}(Q) \xrightarrow{G} Q\text{-}\mathbf{Mod}$, $G(A, e, \bigsqcup) = (A, \leqslant, \bigvee, *)$, $G\varphi = \varphi$, *where*
   - $a_1 \leqslant a_2$ *iff* $1_Q \leqslant e(a_1, a_2)$, *for every* $a_1, a_2 \in A$;
   - $\bigvee S = \bigsqcup \alpha_S^{1_Q}$ *for every* $S \subseteq A$;
   - $q * a = \bigsqcup \alpha_{\{a\}}^{q}$ *for every* $q \in Q$, $a \in A$.
3. $G \circ F = 1_{Q\text{-}\mathbf{Mod}}$ *and* $F \circ G = 1_{\mathbf{Sup}(Q)}$, *i.e., the categories* Q-**Mod** *and* $\mathbf{Sup}(Q)$ *are isomorphic.*

Lattice-valued quantales

Definition 14

- Given a unital quantale Q, a Q-quantale is a tuple $(A, e, \bigsqcup, \otimes)$, in which $(A, e, \bigsqcup)$ is a Q-$\bigvee$-semilattice, and $A \times A \xrightarrow{\otimes} A$ is a map (Q-multiplication on A) such that
  - $(A, \otimes)$ is a semigroup;
  - $a \otimes (\bigsqcup \alpha) = \bigsqcup (a \otimes \cdot)^{\rightarrow}_{Q}(\alpha)$ and $(\bigsqcup \alpha) \otimes a = \bigsqcup (\cdot \otimes a)^{\rightarrow}_{Q}(\alpha)$ for every $a \in A$, $\alpha \in Q^{A}$.
- A Q-quantale homomorphism $(A, e, \bigsqcup, \otimes) \xrightarrow{\varphi} (B, e, \bigsqcup, \otimes)$ is a Q-$\bigvee$-semilattice homomorphism $A \xrightarrow{\varphi} B$ preserving $\otimes$.
- **Quant**(Q) is the category of Q-quantales.

Quantale algebras as lattice-valued quantales

Theorem 15

Let Q be a unital quantale.

1. *There is a functor Q-$\mathbf{Alg} \xrightarrow{F} \mathbf{Quant}(Q)$, $F(A, \otimes, *) = (A, e, \bigsqcup, \otimes)$, $F\varphi = \varphi$, where e and $\bigsqcup$ are from Theorem 13.*
2. *There is a functor $\mathbf{Quant}(Q) \xrightarrow{G} Q$-$\mathbf{Alg}$, $G(A, e, \bigsqcup, \otimes) = (A, \leqslant, \bigvee, *, \otimes)$, $G\varphi = \varphi$, where $\leqslant$, $\bigvee$, $*$ are from Theorem 13.*
3. *$G \circ F = 1_{Q\text{-}\mathbf{Alg}}$ and $F \circ G = 1_{\mathbf{Quant}(Q)}$, i.e., the categories Q-$\mathbf{Alg}$ and $\mathbf{Quant}(Q)$ are isomorphic.*

Lattice-valued quantales versus comma categories

Theorem 16

Given a unital quantale Q, the categories $(Q \downarrow \mathbf{UQuant})_z$, Q-$\mathbf{UAlg}$, and $\mathbf{UQuant}(Q)$ are isomorphic.

Corollary 17

Given a unital quantale Q, the category $(Q \downarrow \mathbf{UQuant})_z$ is isomorphic to a subcategory of $\mathbf{Quant}(Q)$.

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Lattice-valued frames of D. Zhang, Y.-M. Liu and W. Yao

Definition 18

- Given a frame L, L-$\mathbf{UAlg}_{Frm}$ is the full subcategory of L-$\mathbf{UAlg}$, whose objects have frames as their underlying quantales.
- $\mathbf{Frm}(L)$ is the image of the subcategory L-$\mathbf{UAlg}_{Frm}$ under the isomorphism $L\text{-}\mathbf{UAlg} \xrightarrow{F} \mathbf{UQuant}(L)$.

- $\mathbf{Frm}(L)$ is isomorphic to the category of L-frames of W. Yao.

Corollary 19

For a frame L, the categories $(L \downarrow \mathbf{Frm})$ and $\mathbf{Frm}(L)$ are isomorphic.

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Final remarks

- Quantale algebras (motivated by algebras over a not necessarily commutative unital ring) provide a common framework for two notions of lattice-valued frame.
- Fuzzification of the sobriety-spatiality equivalence can be done easier in the setting of quantale algebras.
- Quantale algebras provide a convenient fuzzification of locales.
- Categories **Sup**(Q) and **Quant**(Q) (fuzzy notion) have the properties of the categories Q-**Mod** and Q-**Alg** (crisp notion).

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Thank you for your attention!