On epimorphisms of ordered algebras

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Ordered algebras

- An ordered Ω-algebra is a triple $(\mathcal{A}, \Omega, \leq_A)$,

where $(\mathcal{A}, \Omega)$ is an Ω-algebra,

and $(\mathcal{A}, \leq_A)$ is a poset,

such that every $f^A \in \Omega_A$ is monotone,

i.e. if f^A has arity n, then

- $(x_1 \leq_A x_1' \wedge x_2 \leq_A x_2' \wedge \cdots \wedge x_n \leq_A x_n')$

$$\Longrightarrow f^A(x_1, \ldots, x_n) \leq_A f^A(x_1', \ldots, x_n').$$

- A homomorphism of ordered algebras is a monotone map, that is also homomorphism of the underlaying algebras.

Order-embeddings

- A homomorphism $f : (\mathcal{A}, \Omega, \leq_A) \longrightarrow (\mathcal{B}, \Omega, \leq_B)$ is called an order-embedding if $f(x) \leq_B f(x') \Longrightarrow x \leq_A x'$.
- **Fact** Every order-embedding is injective.
- A surjective order-embedding is called an order-isomorphism.

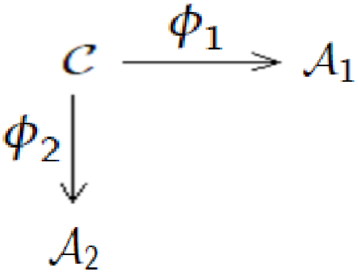
Amalgams of ordered algebras

- An amalgam of ordered algebras is a list $(\mathcal{C}; \mathcal{A}_1, \mathcal{A}_2; \phi_1, \phi_2)$,

where $\mathcal{C}$, $\mathcal{A}_1$ and $\mathcal{A}_2$.are ordered algebras,

and $\phi_i : \mathcal{C} \longrightarrow \mathcal{A}_i$, $i \in \{1, 2\}$, are order-embeddings.

Diagrammatically:

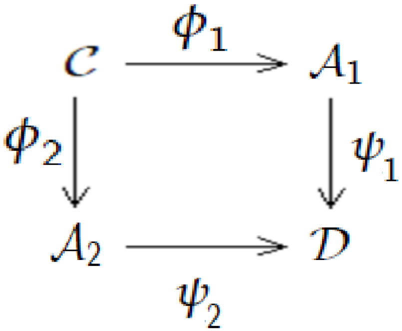


Amalgamation

- We say that $(\mathcal{C}; \mathcal{A}_1, \mathcal{A}_2; \phi_1, \phi_2)$ is **weakly** embeddable if there exists an ordered algebra $\mathcal{D}$,

admitting order-embeddings $\psi_i : \mathcal{A}_i \longrightarrow \mathcal{D}$, $i \in \{1, 2\}$,

such that the following diagram commutes (equivalently, is a pushout):



Amalgamation

- We say that $(\mathcal{C}; \mathcal{A}_1, \mathcal{A}_2; \phi_1, \phi_2)$ is (**strongly**) embeddable if

there exists an ordered algebra $\mathcal{D}$ such that the above diagram is both a pushout and a pullback

that is ψ_1 and ψ_2.agree on $\mathcal{C}$ **only**.

- $(\mathcal{C}; \mathcal{A}_1, \mathcal{A}_2; \phi_1, \phi_2)$ is called a **special amalgam,**

if there exists an order-isomorphism $\nu : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$, with

$\nu \circ \phi_1 = \phi_2$.

- **Fact** Every special amalgam is weakly embeddable.

Special amalgams

- Let $\mathcal{C}$ be an ordered subalgebra of an ordered algebra $\mathcal{A}$.
- Take two disjoint order-isomorphic copies $\mathcal{A}_1$ and $\mathcal{A}_2$ via, say,

$$\nu_i : \mathcal{A} \longrightarrow \mathcal{A}_i, i \in \{1, 2\}.$$

- This gives a special amalgam $(\mathcal{C}; \mathcal{A}_1, \mathcal{A}_2; \nu_1 |_{\mathcal{C}}, \nu_2 |_{\mathcal{C}})$.
- Indeed every special amalgam is the one obtained in this way.
- **Theorem** Let $(\mathcal{C}; \mathcal{A}_1, \mathcal{A}_2; \nu_1 |_{\mathcal{C}}, \nu_2 |_{\mathcal{C}})$ be a special amalgam

that is weakly embedded in $\mathcal{D}$ via $\psi_i : \mathcal{A}_i \longrightarrow \mathcal{D}$ (as noted above).

- Then for $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$, one has

$$\psi_1(a_1) = \psi_2(a_2) \Longrightarrow a_1 = \nu_1(a), \;\; a_2 = \nu_2(a),$$

for some $a \in \mathcal{A}$.

Epis of ordered algebras

- Given a category C of ordered algebras $f \in \mathrm{Mor}(C)$ is termed an epimorphism (shortly epi.) if it is right cancellative,

i.e. for all $g, h \in \mathrm{Mor}(C)$

$$g \circ f = h \circ f \Longrightarrow g = h.$$

- **Fact** Every surjective homomorphism is an epi, but the converse is not true.

Dominions

- Let $\mathcal{C}$ be an ordered subalgebra of an ordered algebra $\mathcal{A}$. Then we define (an ordered subalgebra)

$$\widehat{\text{Dom}}_{\mathcal{A}}\mathcal{C} = \{x \in \mathcal{A} : \forall f, g : \mathcal{A} \longrightarrow \mathcal{B},\ f\mid_{\mathcal{C}} = g\mid_{\mathcal{C}} \implies f(x) = g(x)\}$$

- We call $\widehat{\text{Dom}}_{\mathcal{A}}\mathcal{C}$, the (ordered) dominion of $\mathcal{C}$ in $\mathcal{A}$.
- Treating $\mathcal{C}$ and $\mathcal{A}$ as unordered algebras one gets the analogous definition for $\text{Dom}_{\mathcal{A}}\mathcal{C}$, the unordered dominion of $\mathcal{C}$ in $\mathcal{A}$.

Epimorphisms and dominions

- **Fact**. $\mathcal{C} \subseteq \mathrm{Dom}_{\mathcal{A}}\mathcal{C} \subseteq \widehat{\mathrm{Dom}}_{\mathcal{A}}\mathcal{C} \subseteq \mathcal{A}$.
- **Fact** $f : (\mathcal{A}, \Omega, \leq_A) \longrightarrow (\mathcal{B}, \Omega, \leq_B)$ is an epi iff $\widehat{\mathrm{Dom}}_{\mathcal{B}} \operatorname{Im} f = \mathcal{B}$.
- **Fact** $f : (\mathcal{A}, \Omega) \longrightarrow (\mathcal{B}, \Omega)$ is an epi iff $\mathrm{Dom}_{\mathcal{B}} \operatorname{Im} f = \mathcal{B}$.
- **Consequence** Consider $f : (\mathcal{A}, \Omega, \leq_A) \longrightarrow (\mathcal{B}, \Omega, \leq_B)$.

If $f : (\mathcal{A}, \Omega) \longrightarrow (\mathcal{B}, \Omega)$ is an epi in the unordered context

then so is $f:(\mathcal{A}, \Omega, \leq_A) \longrightarrow (\mathcal{B}, \Omega, \leq_B)$ in the unordered context, for all 'compatible' $\leq_A$ and $\leq_B$ making f monotone.

Epimorphisms and dominions

- **Conjecture 1** (Conversely) If $f : (\mathcal{A}, \Omega, \leq_A) \longrightarrow (\mathcal{B}, \Omega, \leq_B)$ is an epi in the ordered context

then so is $f : (\mathcal{A}, \Omega) \longrightarrow (\mathcal{B}, \Omega)$, in the unordered context.

- **Fact** The conjecture is true in the category of all semigroups (monoids) vs. the category of all ordered semigroups (monoids).
- Clearly, this conjecture will be true if the following is true.
- **Conjecture 2** $\mathrm{Dom}_{\mathcal{A}}\mathcal{C} = \widehat{\mathrm{Dom}}_{\mathcal{A}}\mathcal{C}$.
- **Fact** The above is also true for semigroups (monoids) vs. ordered semigroups (monoids).

Dominions and special amalgamation

- **Fact** We have

$$\widehat{\mathrm{Dom}}_{\mathcal{A}}\mathcal{C} \cong \widehat{\mathrm{Dom}}_{\mathcal{A}_i}\nu_i\,|_{\mathcal{C}}\,(\mathcal{C}) = \psi_i^{-1}\left[\psi_1(\mathcal{A}_1)\cap\psi_2(\mathcal{A}_2)\right],$$

where $\mathcal{D}$ is the 'pushout', (it's a bit abuse of the language).

- **Fact** The analogue of the above holds in the unordered context.
- **Observation** A special amalgam $(\mathcal{C};\mathcal{A}_1,\mathcal{A}_2)$ of ordered (resp. unordered) algebras is embeddable iff

$$\psi_i^{-1}\left[\psi_1(\mathcal{D})\cap\psi_2(\mathcal{D})\right] = \nu_i\,|_{\mathcal{C}}\,(\mathcal{C}),$$

where $\mathcal{D}$ is the respective 'pushout'.

- Hence Conjecture 2 will be true if the following holds,
- **Conjecture 3** A special amalgam $(\mathcal{C};\mathcal{A}_1,\mathcal{A}_2)$ is embeddable in the ordered context iff it is such in the unordered context.

Special amalgamation and epimorphisms

- **Fact** The last conjecture is true for semigroups (monoids) vs. ordered semigroups (monoids).
- **Theorem** Let Ω be a type. Then in the category of all ordered Ω-algebras epis are surjective. (We have a written proof of this.)
- **Theorem** Let Ω be a type. Then in the category of all unordered Ω-algebras epis are surjective.(We have a written proof of this, in fact this is obtained by slightly modifying the above proof.)
- **Corollary** Conjecture 3 is true for any class of all Ω-algebras.

Epimorphisms

- An identity is called **balanced** if in both the terms, that are used to define it, the number of occurrences of every variable is the same.
- **Theorem** Let $\mathcal{V}$ be a variety of Ω- algebras whose defining identities are balanced. Let $\mathcal{V}'$ be the variety of ordered algebras obtained from $\mathcal{V}$. Then Conjecture 3 is true for $\mathcal{V}$ vs. $\mathcal{V}'$. (I don't have a complete written proof but I think I can write one).
- **Question** What about arbitrary $\mathcal{V}$ and $\mathcal{V}'$ (I don't have any proof, or counter example).

References

This research (if it is worth this name) is being conducted jointly with Professor Boza Tasic.

This talk was motivated by the following articles.

[1] Sohail Nasir: Epimorphisms, dominions and amalgamation in pomonoids. Semigroup Forum DOI: 10.1007/s00233-014-9640-x (2014)

[2] Sohail Nasir: Zigzag theorem for partially ordered monoids. Comm. in Algebra 42, 2559–2583 (2014)

[3] Sohail Nasir: Absolute flatness and amalgamation in pomonoids. Semigroup Forum 82 (3), 504–515 (2011)

THANK YOU