Keimel's Problem and threshold convexity

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Gene expression

Protein X *ACTIVATES* production of protein Y

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D
N X binding site RNAp binding site ‖ encoding region for protein Y
A

——————— promotor region ———————

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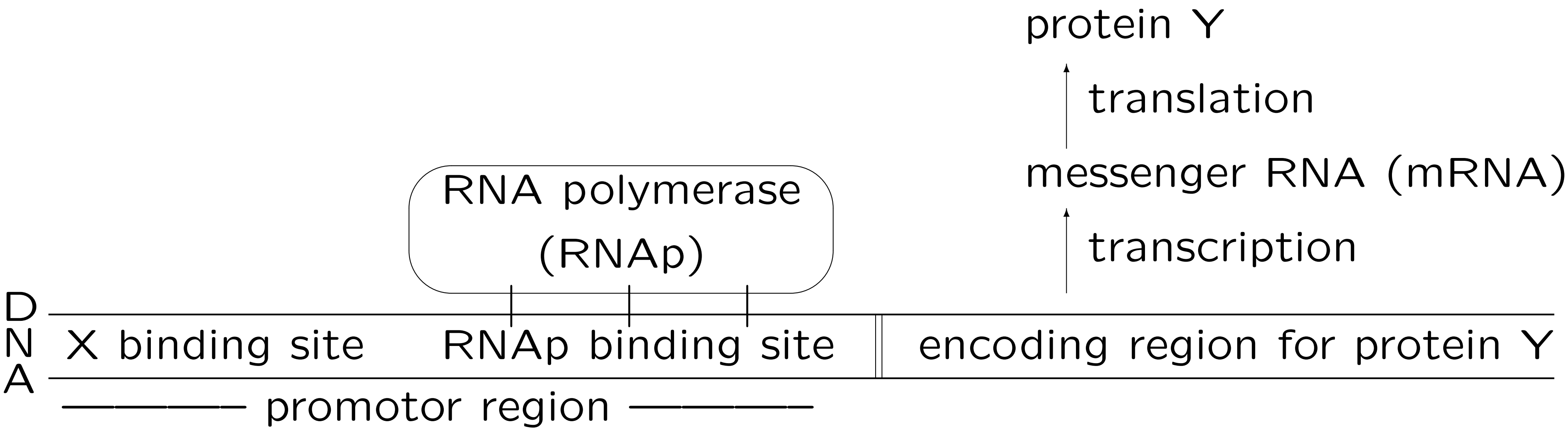

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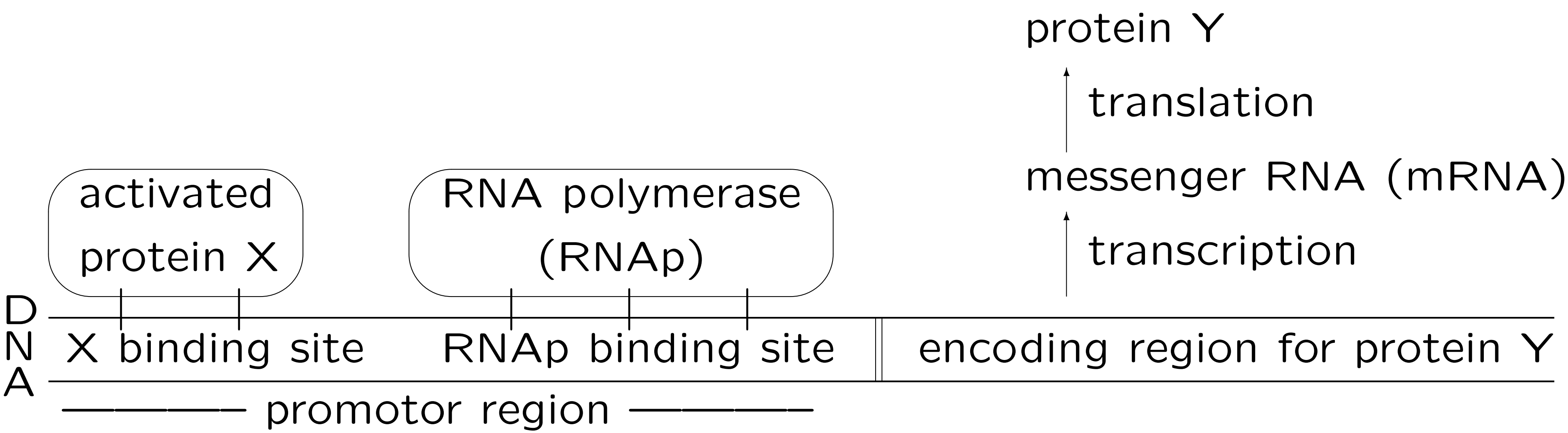
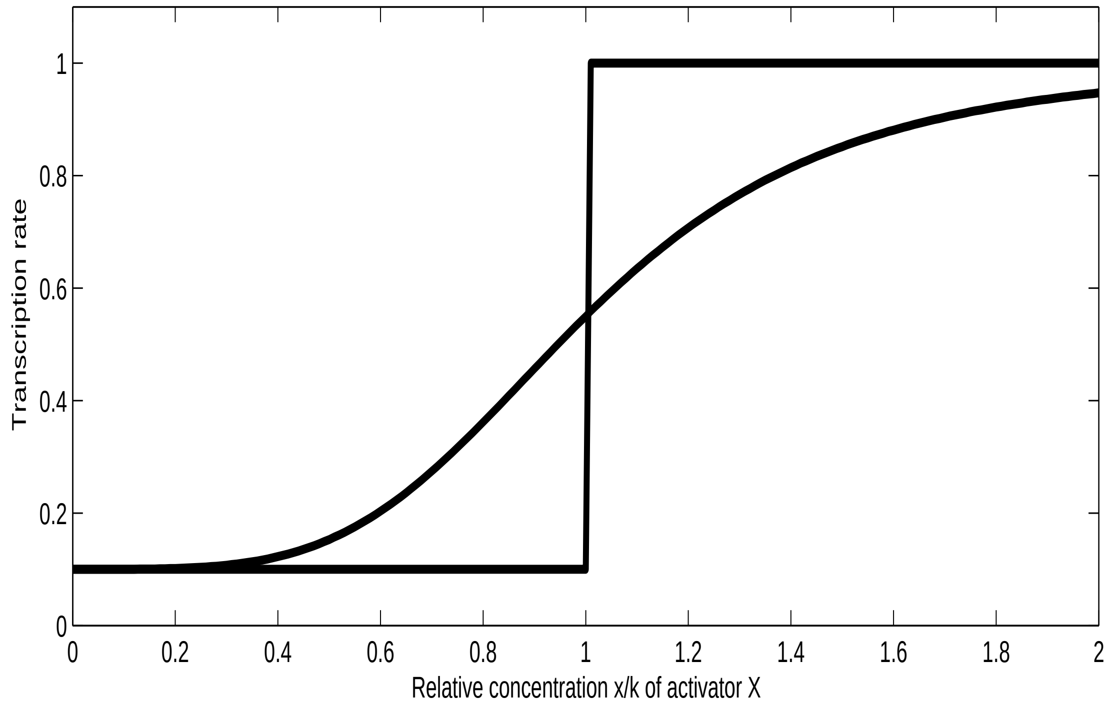
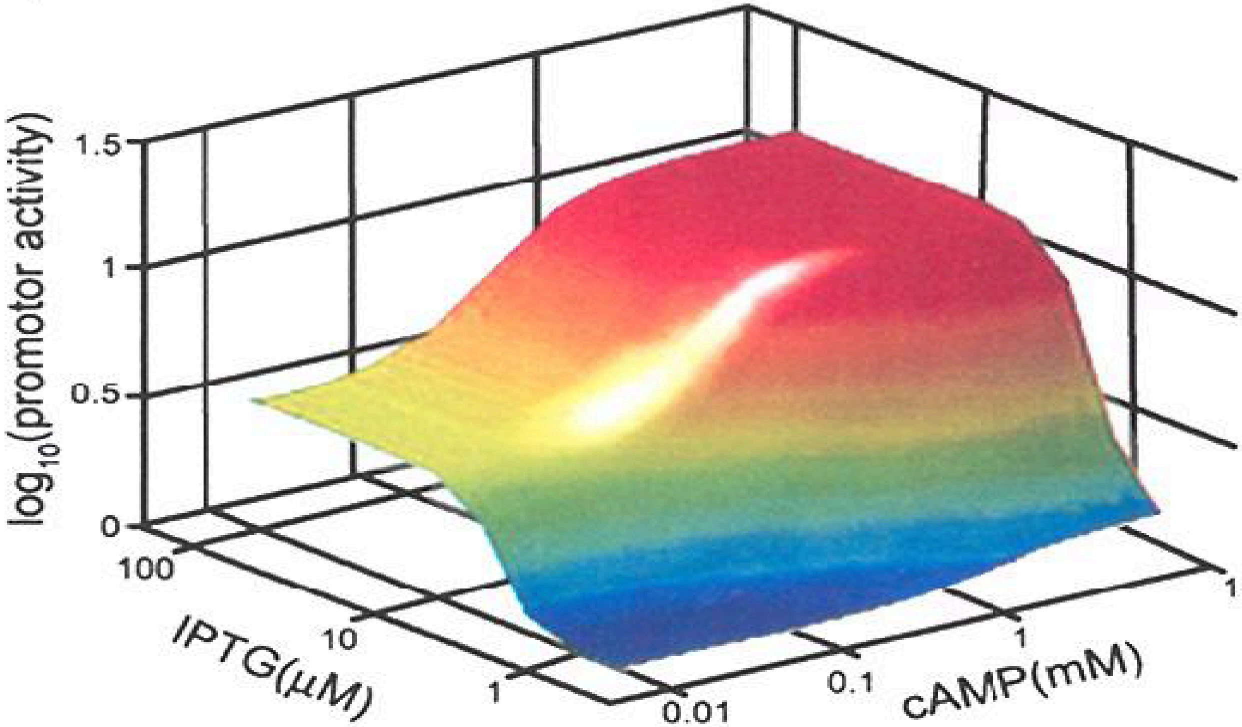


Transcription rate vs. activator X

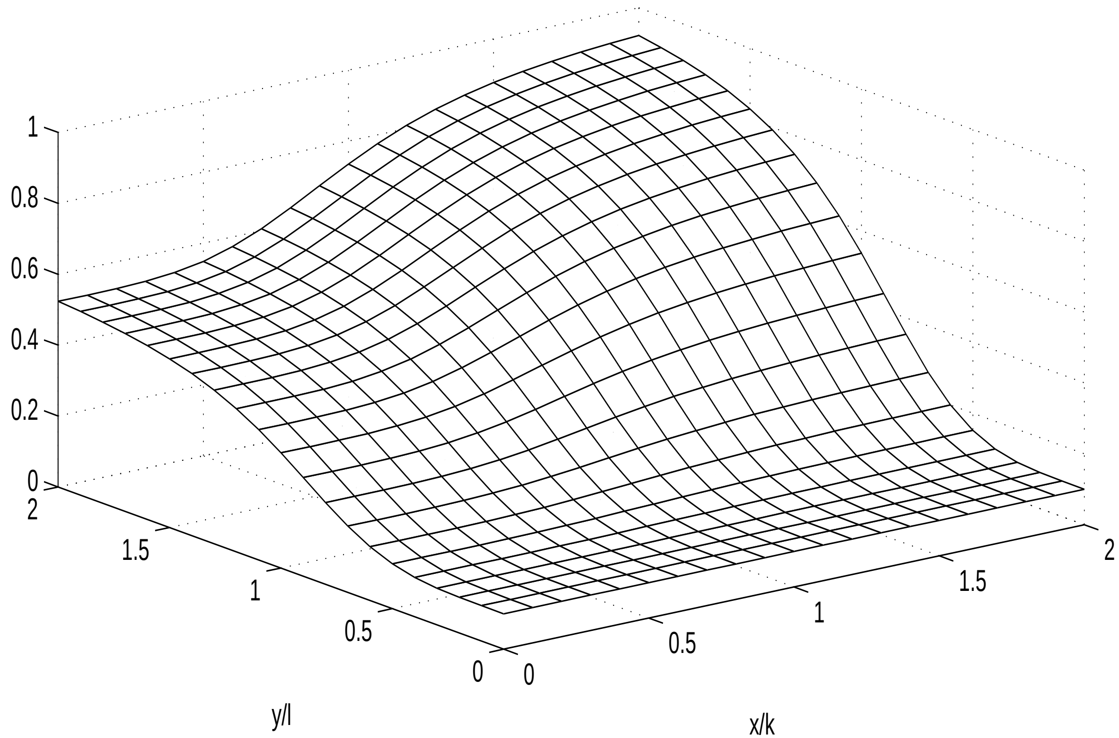


Experimental “and” gate



Fuzzy “and” gate

$$v_0\, v_{01} v_1 \underline{\left[1+\left(\frac{k}{x}\right)^n\right]^{-1}}\ \underline{\left[1+\left(\frac{l}{y}\right)^n\right]^{-1}}$$



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Remark: The open real unit interval

$$I^\circ =]0, 1[= \{p \in \mathbb{R} \mid 0 \leq p \leq 1\}$$

is closed under these three operations.

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Example: Convex sets $(C, \underline{I}^\circ)$, with $xy\underline{p} = x(1-p) + yp$.

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Proposition: A barycentric algebra $(A, \underline{I}^\circ)$ satisfies the **entropic** (hyper-)identity:

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Remark: Recall the **mode** property: idempotence and entropicity,
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Thus a positive answer would axiomatize barycentric algebras
as skew-commutative modes of type $I^\circ \times \{2\}$.

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Note that both idempotence and skew-commutativity hold for the threshold-convex combinations.

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Theorem: $\underline{\underline{B}}^{\frac{1}{2}}$ is the class of commutative, idempotent, entropic magmas.

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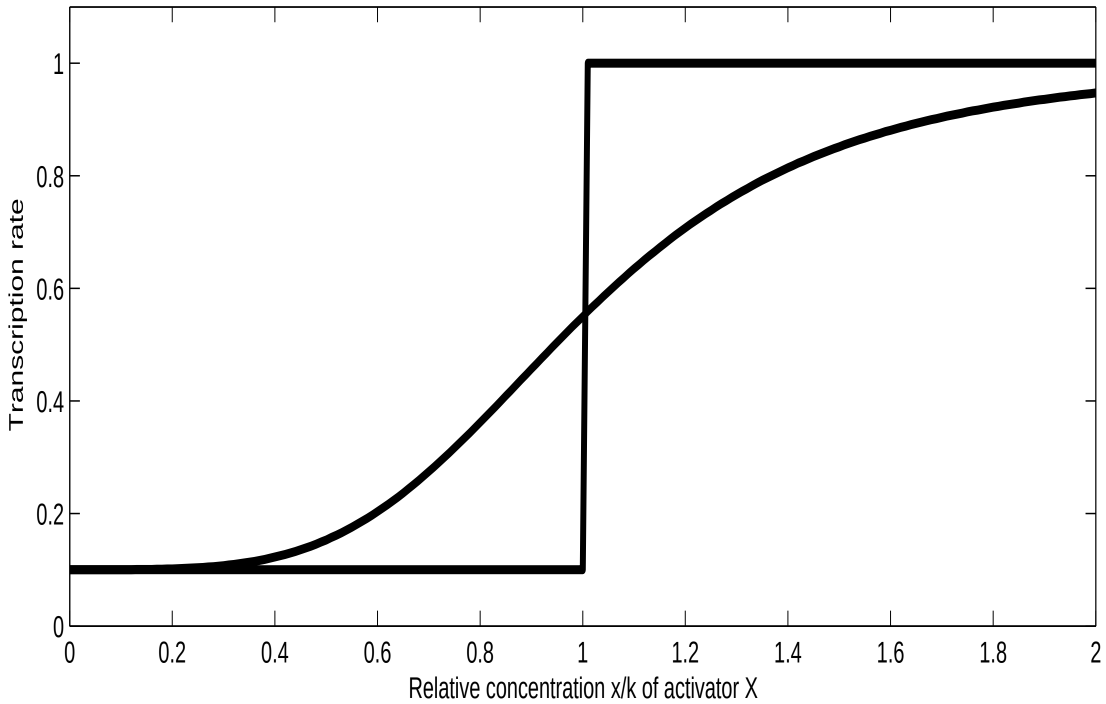
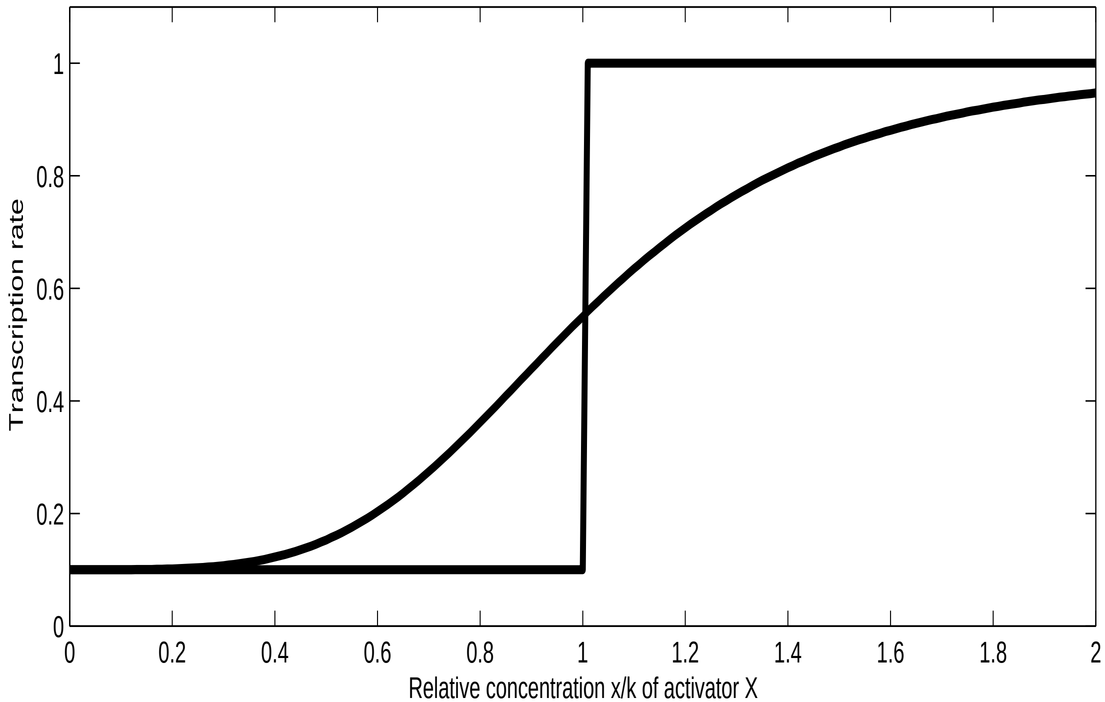


Transcription rate vs. activator X: $t = 0$ and $t = \frac{1}{2}$



Thank you for your attention!