Monoid of Nd-Full Hypersubstitutions

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The aim of this reseach is to shows that the structure

$$(nd\text{-}Hyp^F(\tau_n); \circ_{nd}, \sigma_{id}^{nd}) \text{ is a monoid.}$$

Now we consider algebras of n-ary type, that is, all operation symbols have the same fixed arity n. Let τ_n be such a fixed n-ary type with operation symbols $(f_i)_{i\in I}$ indexed by some set I.

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Definition([1])

Let H_n be the set of all permutations $s : \{1,\ldots,n\} \to \{1,\ldots,n\}$ and let f_i be an operation symbol of type τ_n. Full terms of type τ_n are defined in the following way:

(1) $f_i(x_{s(1)},\ldots,x_{s(n)})$ is a full term of type τ_n.

(2) If $t_1,\ldots,t_n$ are full terms of type τ_n, then $f_i(t_1,\ldots,t_n)$ is a full term of type τ_n.

We denoted by $W^F_{\tau_n}(X_n)$ the set of all full terms of type τ_n.

Example

Let $s : \{1, 2\} \to \{1, 2\}$ and $r : \{1, 2\} \to \{1, 2\}$ which are defined by $s(1) = 2, s(2) = 1$ and $r(1) = 1, r(2) = 2$. Then

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(2) $f(x_{r(1)}, x_{r(2)}) = f(x_1, x_2)$ and

(3) $f(g(x_2, x_1), f(x_1, x_2))$,

are full terms of type $\tau_2 = (2, 2)$.

Definition([1])

Let $W^F_{\tau_n}(X_n)$ be a set of full terms of type τ_n. Then the superposition operations

$$S^n : (W^F_{\tau_n}(X_n))^{n+1} \to W^F_{\tau_n}(X_n),$$

are defined in the following way: For $t, t_q \in W^F_{\tau_n}(X_n), 1 \leq q \leq n, n \in \mathbb{N}$, we have

(1) if $t = f_i(x_{s(1)}, \ldots, x_{s(n)})$ for $s \in H_n$, then $S^n(f_i(x_{s(1)}, \ldots, x_{s(n)}), t_1, \ldots, t_n) := f_i(t_{s(1)}, \ldots, t_{s(n)})$,

(2) if $t = f_i(s_1, \ldots, s_n)$ and if we assume that $S^n(s_q, t_1, \ldots, t_n)$ are already defined, then $S^n(f_i(s_1, \ldots, s_n), t_1, \ldots, t_n) :=$ $f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_n, t_1, \ldots, t_n))$.

For a full term t we need the full term t_s arising from t by replacement a variable $x_i, 1 \leq i \leq n$ in t by a variable $x_{s(i)}$ for a permutation $s \in H_n$. This can be defined as follows:

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(1) If $t = f_i(x_{r(1)}, \ldots, x_{r(n)})$, then $t_s := f_i(x_{s(r(1))}, \ldots, x_{s(r(n))})$.

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Let $s : \{1, 2\} \to \{1, 2\}$ and $r : \{1, 2\} \to \{1, 2\}$ which are defined by $s(1) = 2, s(2) = 1$ and $r(1) = 1, r(2) = 2$. Let $t = f(g(x_{s(1)}, x_{s(2)}), f(x_{r(1)}, x_{r(2)}))$. Then

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$$\begin{aligned} t_s &= (f(g(x_{s(1)}, x_{s(2)}), f(x_{r(1)}, x_{r(2)})))_s \\ &= (f(g(x_2, x_1), f(x_1, x_2)))_s \end{aligned}$$

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Let $W_{\tau_n}^F(X_n)$ be a set of all full terms of type τ_n and let T be a subset of $W_{\tau_n}^F(X_n)$ and $s \in H_n$. Then we set

$$T_s := \begin{cases} \{t_s \mid t \in W_{\tau_n}^F(X_n)\} & \text{if } T \neq \emptyset \\ \emptyset & \text{if } T = \emptyset. \end{cases}$$

Definition

Let $W^F_{\tau_n}(X_n)$ be a set of all full terms of type τ_n and $s \in H_n$. Then the superposition operations

$$S^n_{nd} : (\mathcal{P}(W^F_{\tau_n}(X_n)))^{n+1} \to \mathcal{P}(W^F_{\tau_n}(X_n)),$$

for $T, T_q \subseteq W^F_{\tau_n}(X_n), 1 \leq q \leq n, n \in \mathbb{N}$ such that T, T_q are non-empty sets, the $S^n_{nd}(T, T_1, \ldots, T_n)$ are defined in the following way:

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(1) If $T = \{f_i(x_{s(1)}, \ldots, x_{s(n)})\}$, then $S^n_{nd}(\{f_i(x_{s(1)}, \ldots, x_{s(n)})\}, T_1, \ldots, T_n) := \{f_i(t_{s(1)}, \ldots, t_{s(n)}) \mid t_{s(q)} \in T_{s(q)}\}$.

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(2) If $T = \{f_i(t_1, \ldots, t_n)\}$, then $S^n_{nd}(\{f_i(t_1, \ldots, t_n)\}, T_1, \ldots, T_n) := \{f_i(r_1, \ldots, r_n) \mid r_q \in S^n_{nd}(\{t_q\}, T_1, \ldots, T_n)\}$.

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(2) If $T = \{f_i(t_1, \ldots, t_n)\}$, then $S^n_{nd}(\{f_i(t_1, \ldots, t_n)\}, T_1, \ldots, T_n) := \{f_i(r_1, \ldots, r_n) \mid r_q \in S^n_{nd}(\{t_q\}, T_1, \ldots, T_n)\}$.

(3) If T is an arbitrary subset of $W^F_{\tau_n}(X_n)$, then $S^n_{nd}(T, T_1, \ldots, T_n) := \bigcup S^n_{nd}(\{t\}, T_1, \ldots, T_n)$.

Definition (continuous)

If one of the sets $T, T_1, \ldots, T_n$ is an empty set, then $S^n_{nd}(T, T_1, \ldots, T_n) := \emptyset$.

Example

Let $s : \{1, 2\} \to \{1, 2\}$ and $r : \{1, 2\} \to \{1, 2\}$ which are defined by $s(1) = 2, s(2) = 1$ and $r(1) = 1, r(2) = 2$. Let $T = \{g(x_{s(1)}, x_{s(2)}), f(x_{r(1)}, x_{r(2)})\}$, $T_1 = \{f(x_{r(1)}, x_{r(2)})\}$ and $T_2 = \{g(x_{s(1)}, x_{s(2)})\}$. Then we have

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$$\begin{aligned} S^2_{nd}(\{g(x_{s(1)}, x_{s(2)})\}, T_1, T_2) &= S^2_{nd}(\{g(x_2, x_1)\}, T_1, T_2) \\ &= \{g(v_2, v_1) \mid v_2 \in T_2, v_1 \in T_1\} \\ &= \{g(g(x_{s(1)}, x_{s(2)}), f(x_{r(1)}, x_{r(2)}))\} \\ &= \{g(g(x_2, x_1), f(x_1, x_2))\} \text{ and,} \end{aligned}$$

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$$\begin{aligned}
S^2_{nd}(\{g(x_{s(1)}, x_{s(2)})\}, T_1, T_2) &= S^2_{nd}(\{g(x_2, x_1)\}, T_1, T_2) \\
&= \{g(v_2, v_1) \mid v_2 \in T_2, v_1 \in T_1\} \\
&= \{g(g(x_{s(1)}, x_{s(2)}), f(x_{r(1)}, x_{r(2)}))\} \\
&= \{g(g(x_2, x_1), f(x_1, x_2))\} \text{ and,} \\
S^2_{nd}(\{f(x_{r(1)}, x_{r(2)})\}, T_1, T_2) &= S^2_{nd}(\{f(x_1, x_2)\}, T_1, T_2) \\
&= \{f(u_1, u_2) \mid u_1 \in T_1, u_2 \in T_2\} \\
&= \{f(f(x_{r(1)}, x_{r(2)}), g(x_{s(1)}, x_{s(2)}))\} \\
&= \{f(f(x_1, x_2), g(x_2, x_1))\}.
\end{aligned}$$

Therefore we have

Example (Continuous)

$$
\begin{aligned}
S^2_{nd}(T, T_1, T_2) &= S^2_{nd}(\{g(x_{s(1)}, x_{s(2)}), f(x_{r(1)}, x_{r(2)})\}, T_1, T_2) \\
&= S^2_{nd}(\{g(x_2, x_1), f(x_1, x_2)\}, T_1, T_2) \\
&= S^2_{nd}(\{g(x_2, x_1)\}, T_1, T_2) \bigcup S^2_{nd}(\{f(x_1, x_2)\}, T_1, T_2) \\
&= \{g(g(x_2, x_1), f(x_1, x_2))\} \bigcup \{f(f(x_1, x_2), g(x_2, x_1))\} \\
&= \{g(g(x_2, x_1), f(x_1, x_2)), f(f(x_1, x_2), g(x_2, x_1))\}.
\end{aligned}
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Proposition 1

Let $T, T_q \subseteq W^F_{\tau_n}(X_n), 1 \leq q \leq n, n \in \mathbb{N}$ and $s \in H_n$. Then we have

(1) $S^n_{nd}(T_s, T_1, \ldots, T_n) = S^n_{nd}(T, T_{s(1)}, \ldots, T_{s(n)})$.

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(1) $S^n_{nd}(T_s, T_1, \ldots, T_n) = S^n_{nd}(T, T_{s(1)}, \ldots, T_{s(n)})$.

(2) $S^n_{nd}(T_s, T_1, \ldots, T_n) = (S^n_{nd}(T, T_1, \ldots, T_n))_s$.

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Proposition 2

Let $T, T_q \subseteq W^F_{\tau_n}(X_n), 1 \leq q \leq n, n \in \mathbb{N}$ and $s \in H_n$. Then we have

$$S^n_{nd}(T, T_{s(1)}, \ldots, T_{s(n)}) = (S^n_{nd}(T, T_1, \ldots, T_n))_s.$$

Theorem 3

Let $T, T_q, S_q \subseteq W^F_{\tau_n}(X_n), 1 \leq q \leq n, n \in \mathbb{N}$ be the set of full terms of type τ_n. Then we have

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$S_{nd}^n(T, S_{nd}^n(S_1, T_1, \ldots, T_n), \ldots, S_{nd}^n(S_n, T_1, \ldots, T_n))$
$= S_{nd}^n(S_{nd}^n(T, S_1, \ldots, S_n), T_1, \ldots, T_n)$

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Using this superposition operation we can forms algebra

$$(\mathcal{P}(W^F_{\tau_n}(X_n)); S^n_{nd})$$

of type $n + 1$.

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Using this superposition operation we can forms algebra

$$(\mathcal{P}(W^F_{\tau_n}(X_n)); S^n_{nd})$$

of type $n+1$.This algebra is called

$$nd\text{-}clone_F \tau_n.$$

Definition

A mapping $\sigma^{nd} : \{f_i \mid i \in I\} \to \mathcal{P}(W^F_{\tau_n}(X_n))$ is called non-deterministic full hypersubstitution or nd-full hypersubstitution, for short. Let $nd\text{-}Hyp^F(\tau_n)$ be a set of all nd-full hypersubstitutions. Any such nd-full hypersubstitution, σ^{nd} uniquely determine a mapping

$$\hat{\sigma}^{nd} : \mathcal{P}(W^F_{\tau_n}(X_n)) \to \mathcal{P}(W^F_{\tau_n}(X_n)),$$

is defined in the following way:

Definition (Continuous)

(1) $\hat{\sigma}^{nd}[\emptyset] := \emptyset.$

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(2) $\hat{\sigma}^{nd}[\{f_i(x_{s(1)}, \ldots, x_{s(n)})\}] := (\sigma^{nd}(f_i))_s$ for every $s \in H_n$.

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(4) $\hat{\sigma}^{nd}[T] := \bigcup_{t \in T} \hat{\sigma}^{nd}[\{t\}]$ where T is an arbitrary subset of $W^F_{\tau_n}(X_n)$.

Example

Let $s : \{1,2\} \to \{1,2\}$ and $r : \{1,2\} \to \{1,2\}$ which are defined by $s(1) = 2, s(2) = 1$ and $r(1) = 1, r(2) = 2$.
Let $T = \{g(f(x_{r(1)}, x_{r(2)}), g(x_{s(1)}, x_{s(2)})), f(x_{r(1)}, x_{r(2)})\}$, and let $\sigma^{nd} : \{g, f\} \to \mathcal{P}(W_{\tau_2}^F(X_2))$ be defined by
$\sigma^{nd}(g) := \{f(x_{r(1)}, x_{r(2)})\}$,$\sigma^{nd}(f) := \{g(x_{s(1)}, x_{s(2)})\}$. Then we have

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$\sigma^{nd}(g):=\{f(x_{r(1)},x_{r(2)})\}$,$\sigma^{nd}(f):=\{g(x_{s(1)},x_{s(2)})\}$. Then we have

$$\hat{\sigma}^{nd}(T) \quad = \quad \hat{\sigma}^{nd}(\{g(f(x_{r(1)},x_{r(2)}),g(x_{s(1)},x_{s(2)})),f(x_{r(1)},x_{r(2)})\})$$

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$\sigma^{nd}(g) := \{f(x_{r(1)}, x_{r(2)})\}$,$\sigma^{nd}(f) := \{g(x_{s(1)}, x_{s(2)})\}$. Then we have

$$\begin{aligned} \hat{\sigma}^{nd}(T) &= \hat{\sigma}^{nd}(\{g(f(x_{r(1)}, x_{r(2)}), g(x_{s(1)}, x_{s(2)})), f(x_{r(1)}, x_{r(2)})\}) \\ &= \hat{\sigma}^{nd}(\{g(f(x_{r(1)}, x_{r(2)}), g(x_{s(1)}, x_{s(2)}))\} \bigcup \\ &\quad \hat{\sigma}^{nd}(\{f(x_{r(1)}, x_{r(2)})\}). \end{aligned}$$

Example (Continuous)

Let us consider the following equations:

$$\hat{\sigma}^{nd}(\{g(f(x_{r(1)}, x_{r(2)}), g(x_{s(1)}, x_{s(2)}))\}$$
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Let us consider the following equations:

$$\begin{aligned}
&\hat{\sigma}^{nd}(\{g(f(x_{r(1)}, x_{r(2)}), g(x_{s(1)}, x_{s(2)}))\} \\
&= S^2_{nd}(\sigma^{nd}(g), \hat{\sigma}^{nd}(\{f(x_{r(1)}, x_{r(2)})\}), \hat{\sigma}^{nd}(\{g(x_{s(1)}, x_{s(2)})\})) \\
&= S^2_{nd}(\sigma^{nd}(g), (\sigma^{nd}(f))_r, (\sigma^{nd}(g))_s) \\
&= S^2_{nd}(\sigma^{nd}(g), (\{g(x_{s(1)}, x_{s(2)})\})_r, (\{f(x_{r(1)}, x_{r(2)})\})_s) \\
&= S^2_{nd}(\sigma^{nd}(g), (\{g(x_2, x_1)\})_r, (\{f(x_1, x_2)\})_s) \\
&= S^2_{nd}(\sigma^{nd}(g), \{g(x_{r(2)}, x_{r(1)})\}, \{f(x_{s(1)}, x_{s(2)})\}) \\
&= S^2_{nd}(\{f(x_{r(1)}, x_{r(2)})\}, \{g(x_2, x_1)\}, \{f(x_2, x_1)\}) \\
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\end{aligned}$$

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Let us consider the following equations:

$$\hat{\sigma}^{nd}(\{g(f(x_{r(1)},x_{r(2)}),g(x_{s(1)},x_{s(2)}))\}$$
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$$= \{f(r_1,r_2) \mid r_1 \in \{g(x_2,x_1)\}, r_2 \in \{f(x_2,x_1)\}\}$$
$$= \{f(g(x_2,x_1),f(x_2,x_1))\} \text{ and}$$

Example (Continuous)

$$\hat{\sigma}^{nd}(\{f(x_{r(1)}, x_{r(2)})\} \quad = \quad (\sigma^{nd}(f))_r$$

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$$\begin{aligned}\hat{\sigma}^{nd}(\{f(x_{r(1)}, x_{r(2)})\} &= (\sigma^{nd}(f))_r \\ &= (\{g(x_{s(1)}, x_{s(2)})\})_r\end{aligned}$$

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\hat{\sigma}^{nd}(\{f(x_{r(1)}, x_{r(2)})\} &= (\sigma^{nd}(f))_r \\
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Therefore we have that

$$\hat{\sigma}^{nd}(T) = \{f(g(x_2, x_1), f(x_2, x_1))\} \bigcup \{g(x_2, x_1)\}$$

Example (Continuous)

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&= \{f(g(x_2, x_1), f(x_2, x_1)), g(x_2, x_1)\}.
\end{aligned}
$$

Lemma 4

Let T be a subset of $W^F_{\tau_n}(X_n)$ and $s \in H_n$. Then we have

$$\hat{\sigma}^{nd}[T_s] = (\hat{\sigma}^{nd}[T])_s.$$

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Theorem 5

A mapping $\hat{\sigma}^{nd} : \mathcal{P}(W^F_{\tau_n}(X_n)) \to \mathcal{P}(W^F_{\tau_n}(X_n))$ is an endomorphism of $nd\text{-}clone_F\tau_n$.

Let $\sigma_1^{nd}, \sigma_2^{nd} \in nd\text{-}Hyp^F(\tau_n)$.

Let $\sigma_1^{nd}, \sigma_2^{nd} \in nd\text{-}Hyp^F(\tau_n)$. Since the extension of non-derministic full hypersubstitution maps $\mathcal{P}(W_{\tau_n}^F(X_n))$ to $\mathcal{P}(W_{\tau_n}^F(X_n))$ we may define a product $\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}$ by

$$\sigma_1^{nd} \circ_{nd} \sigma_2^{nd} := \hat{\sigma}_1^{nd} \circ \sigma_2^{nd}.$$

Here $\circ$ is the usual composition of mappings. Since $\hat{\sigma}_1^{nd} \circ \sigma_2^{nd}$ maps $\{f_i \mid i \in I\}$ to $\mathcal{P}(W_{\tau_n}^F(X_n))$, it is a non-derterministic full hypersubstitution.

Lemma 6

Let $\sigma_1^{nd}, \sigma_2^{nd} \in nd\text{-}Hyp^F(\tau_n)$. Then we have

$$(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd})^\wedge = \hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd}.$$

Lemma 6

Let $\sigma_1^{nd}, \sigma_2^{nd} \in nd\text{-}Hyp^F(\tau_n)$. Then we have

$$(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd})\hat{} = \hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd}.$$

Lemma 7

The binary operation $\circ_{nd}$ is associative.

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Lemma 7

The binary operation $\circ_{nd}$ is associative.

Let $\sigma_{id}^{nd} \in nd\text{-}Hyp^F(\tau_n)$. We define $\sigma_{id}^{nd}(f_i) := \{f_i(x_1, \ldots, x_n)\}$ and the next lemma we show that the extension of σ_{id}^{nd} is an identity mapping.

Lemma 8

Let $T \subseteq W_{\tau_n}^F(X_n)$ be a subset of $W_{\tau_n}^F(X_n)$. Then we have

$$\hat{\sigma}_{id}^{nd}[T] = T.$$

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$$\hat{\sigma}_{id}^{nd}[T] = T.$$

Lemma 9

The σ_{id}^{nd} in $nd\text{-}Hyp^F(\tau_n)$ is an identity element in the set $nd\text{-}Hyp^F(\tau_n)$ with respect to the associative binary operation $\circ_{nd}$.

Lemma 8

Let $T \subseteq W^F_{\tau_n}(X_n)$ be a subset of $W^F_{\tau_n}(X_n)$. Then we have

$$\hat{\sigma}^{nd}_{id}[T] = T.$$

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The σ^{nd}_{id} in $nd\text{-}Hyp^F(\tau_n)$ is an identity element in the set $nd\text{-}Hyp^F(\tau_n)$ with respect to the associative binary operation $\circ_{nd}$.

Theorem 10

The structure $(nd\text{-}Hyp^F(\tau_n); \circ_{nd}, \sigma^{nd}_{id})$ is a monoid.

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Thank You For Yours Attentions