The Cycle Structure of Quandles

Naqeeb ur Rehman

Philipps University Marburg
(Joint work with István Heckenberger)

Problem and References

Problem and References

Problem: Hayashi's conjecture on the cycle structure of quandles.

Problem and References

Problem: Hayashi's conjecture on the cycle structure of quandles.

References:

Problem and References

Problem: Hayashi's conjecture on the cycle structure of quandles.

References:

- P. Lopes and D. Roseman, *On finite racks and quandles,* Comm. Algebra., (2006).

Problem and References

Problem: Hayashi's conjecture on the cycle structure of quandles.

References:

- P. Lopes and D. Roseman, *On finite racks and quandles,* Comm. Algebra., (2006).
- C. Hayashi, *Canonical forms for operation tables of finite connected quandles,* Comm. Algebra., (2013).

Problem and References

Problem: Hayashi's conjecture on the cycle structure of quandles.

References:

- P. Lopes and D. Roseman, *On finite racks and quandles,* Comm. Algebra., (2006).
- C. Hayashi, *Canonical forms for operation tables of finite connected quandles,* Comm. Algebra., (2013).
- A. Bors, *On finite groups where the order of every automorphism is a cycle length,* arxiv 1412.8418., (2014).

Problem and References

Problem: Hayashi's conjecture on the cycle structure of quandles.

References:

- P. Lopes and D. Roseman, *On finite racks and quandles,* Comm. Algebra., (2006).
- C. Hayashi, *Canonical forms for operation tables of finite connected quandles,* Comm. Algebra., (2013).
- A. Bors, *On finite groups where the order of every automorphism is a cycle length,* arxiv 1412.8418., (2014).
- I. Heckenberger, A. Lochmann, and L. Vendramin, *Nichols algebras with many cubic relations,* Trans. Amer. Math. Soc., (2015).

Problem and References

Problem: Hayashi's conjecture on the cycle structure of quandles.

References:

- P. Lopes and D. Roseman, *On finite racks and quandles,* Comm. Algebra., (2006).
- C. Hayashi, *Canonical forms for operation tables of finite connected quandles,* Comm. Algebra., (2013).
- A. Bors, *On finite groups where the order of every automorphism is a cycle length,* arxiv 1412.8418., (2014).
- I. Heckenberger, A. Lochmann, and L. Vendramin, *Nichols algebras with many cubic relations,* Trans. Amer. Math. Soc., (2015).
- S. Guest and P. Spiga, *Finite primitive groups and regular orbits of group elements,* Trans. Amer. Math. Soc., (2016).

Definitions

A **rack** is a pair $(X, \triangleright)$, where X is a non-empty set and $\triangleright : X \times X \longrightarrow X$ is a binary operation, such that for all $x, y, z \in X$:

Definitions

A **rack** is a pair $(X, \triangleright)$, where X is a non-empty set and $\triangleright : X \times X \longrightarrow X$ is a binary operation, such that for all $x, y, z \in X$:

- the map $\varphi_x : X \longrightarrow X$, $\varphi_x(y) := x \triangleright y$, is a bijection,

Definitions

A **rack** is a pair $(X, \triangleright)$, where X is a non-empty set and $\triangleright : X \times X \longrightarrow X$ is a binary operation, such that for all $x, y, z \in X$:

- the map $\varphi_x : X \longrightarrow X$, $\varphi_x(y) := x \triangleright y$, is a bijection,
- $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ (i.e., $\triangleright$ is self-distributive).

Definitions

A **rack** is a pair $(X, \triangleright)$, where X is a non-empty set and $\triangleright : X \times X \longrightarrow X$ is a binary operation, such that for all $x, y, z \in X$:

- the map $\varphi_x : X \longrightarrow X$, $\varphi_x(y) := x \triangleright y$, is a bijection,
- $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ (i.e., $\triangleright$ is self-distributive).

A **quandle** is a rack $(X, \triangleright)$ which further satisfies

- $x \triangleright x = x$ for all $x \in X$ (i.e., $\triangleright$ is idempotent).

Definitions

A **rack** is a pair $(X, \triangleright)$, where X is a non-empty set and $\triangleright : X \times X \longrightarrow X$ is a binary operation, such that for all $x, y, z \in X$:

- the map $\varphi_x : X \longrightarrow X$, $\varphi_x(y) := x \triangleright y$, is a bijection,
- $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ (i.e., $\triangleright$ is self-distributive).

A **quandle** is a rack $(X, \triangleright)$ which further satisfies

- $x \triangleright x = x$ for all $x \in X$ (i.e., $\triangleright$ is idempotent).

A **crossed set** is a quandle $(X, \triangleright)$ which further satisfies

- $x \triangleright y = y$ whenever $y \triangleright x = x$.

Definitions

A **rack** is a pair $(X, \triangleright)$, where X is a non-empty set and $\triangleright : X \times X \longrightarrow X$ is a binary operation, such that for all $x, y, z \in X$:

- the map $\varphi_x : X \longrightarrow X$, $\varphi_x(y) := x \triangleright y$, is a bijection,
- $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ (i.e., $\triangleright$ is self-distributive).

A **quandle** is a rack $(X, \triangleright)$ which further satisfies

- $x \triangleright x = x$ for all $x \in X$ (i.e., $\triangleright$ is idempotent).

A **crossed set** is a quandle $(X, \triangleright)$ which further satisfies

- $x \triangleright y = y$ whenever $y \triangleright x = x$.

The **inner group** of a rack X is the group

$$Inn(X) = \langle \varphi_x | x \in X \rangle.$$

Definitions

A **rack** is a pair $(X, \triangleright)$, where X is a non-empty set and $\triangleright : X \times X \longrightarrow X$ is a binary operation, such that for all $x, y, z \in X$:

- the map $\varphi_x : X \longrightarrow X$, $\varphi_x(y) := x \triangleright y$, is a bijection,
- $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ (i.e., $\triangleright$ is self-distributive).

A **quandle** is a rack $(X, \triangleright)$ which further satisfies

- $x \triangleright x = x$ for all $x \in X$ (i.e., $\triangleright$ is idempotent).

A **crossed set** is a quandle $(X, \triangleright)$ which further satisfies

- $x \triangleright y = y$ whenever $y \triangleright x = x$.

The **inner group** of a rack X is the group

$$Inn(X) = \langle \varphi_x | x \in X \rangle.$$

A rack X is called **indecomposable or connected** if $Inn(X)$ acts transitively on X.

Examples

Examples

- **Conjugation Quandle** on a group G with $x \triangleright y = xyx^{-1}$.

Examples

- **Conjugation Quandle** on a group G with $x \triangleright y = xyx^{-1}$. In particular, the union of conjugacy classes of G is also a quandle under conjugation.

Examples

- **Conjugation Quandle** on a group G with $x \triangleright y = xyx^{-1}$. In particular, the union of conjugacy classes of G is also a quandle under conjugation.
- **Dihedral Quandle** $\mathbb{D}_n$ on $\mathbb{Z}_n$ with $i \triangleright j = 2j - i \pmod{n}$.

Examples

- **Conjugation Quandle** on a group G with $x \triangleright y = xyx^{-1}$. In particular, the union of conjugacy classes of G is also a quandle under conjugation.
- **Dihedral Quandle** $\mathbb{D}_n$ on $\mathbb{Z}_n$ with $i \triangleright j = 2j - i \pmod{n}$.
- Let A be an abelian group, $\alpha \in Aut(A)$ and $1 = id_A$. Then we have a quandle structure on A, defined by:

$$x \triangleright y = (1 - \alpha)(x) + \alpha(y).$$

This quandle is known as **affine quandle** $Aff(A, \alpha)$.

Examples

- **Conjugation Quandle** on a group G with $x \triangleright y = xyx^{-1}$. In particular, the union of conjugacy classes of G is also a quandle under conjugation.
- **Dihedral Quandle** $\mathbb{D}_n$ on $\mathbb{Z}_n$ with $i \triangleright j = 2j - i \pmod{n}$.
- Let A be an abelian group, $\alpha \in Aut(A)$ and $1 = id_A$. Then we have a quandle structure on A, defined by:
$$x \triangleright y = (1 - \alpha)(x) + \alpha(y).$$
This quandle is known as **affine quandle** $Aff(A, \alpha)$.
- Let G be a group, $\alpha \in Aut(G)$, and H be a subgroup of the fixed points of α in G. Then for any $g, f \in G$, the quandle structure on G/H is defined by:
$$gH \triangleright fH = g\alpha(g^{-1}f)H.$$
This quandle is known as **coset quandle** (G, H, α).

Definition

Definition

Let X be a finite rack and $\varphi_x \in Aut(X)$.

Definition

Let X be a finite rack and $\varphi_x \in Aut(X)$. Let $\varphi_x = \sigma_1\sigma_2...\sigma_k$ be the decomposition of φ_x into product of disjoint cycles σ_i.

Definition

Let X be a finite rack and $\varphi_x \in Aut(X)$. Let $\varphi_x = \sigma_1\sigma_2...\sigma_k$ be the decomposition of φ_x into product of disjoint cycles σ_i. Let $\ell(\sigma_i) = \ell_i$ is the cycle length of σ_i and $1 \leq \ell_1 \leq \ell_2 \leq ... \leq \ell_k$.

Definition

Let X be a finite rack and $\varphi_x \in Aut(X)$. Let $\varphi_x = \sigma_1\sigma_2...\sigma_k$ be the decomposition of φ_x into product of disjoint cycles σ_i. Let $\ell(\sigma_i) = \ell_i$ is the cycle length of σ_i and $1 \leq \ell_1 \leq \ell_2 \leq ... \leq \ell_k$. Then the list of all ℓ_i is called the **pattern** of φ_x.

Definition

Let X be a finite rack and $\varphi_x \in Aut(X)$. Let $\varphi_x = \sigma_1\sigma_2...\sigma_k$ be the decomposition of φ_x into product of disjoint cycles σ_i. Let $\ell(\sigma_i) = \ell_i$ is the cycle length of σ_i and $1 \leq \ell_1 \leq \ell_2 \leq ... \leq \ell_k$. Then the list of all ℓ_i is called the **pattern** of φ_x. The **profile** of X is the sequence of patterns of all $\varphi_x \in Aut(X)$.

Definition

Let X be a finite rack and $\varphi_x \in Aut(X)$. Let $\varphi_x = \sigma_1\sigma_2 ... \sigma_k$ be the decomposition of φ_x into product of disjoint cycles σ_i. Let $\ell(\sigma_i) = \ell_i$ is the cycle length of σ_i and $1 \leq \ell_1 \leq \ell_2 \leq ... \leq \ell_k$. Then the list of all ℓ_i is called the **pattern** of φ_x. The **profile** of X is the sequence of patterns of all $\varphi_x \in Aut(X)$.

Remarks

Definition

Let X be a finite rack and $\varphi_x \in Aut(X)$. Let $\varphi_x = \sigma_1\sigma_2...\sigma_k$ be the decomposition of φ_x into product of disjoint cycles σ_i. Let $\ell(\sigma_i) = \ell_i$ is the cycle length of σ_i and $1 \leq \ell_1 \leq \ell_2 \leq ... \leq \ell_k$. Then the list of all ℓ_i is called the **pattern** of φ_x. The **profile** of X is the sequence of patterns of all $\varphi_x \in Aut(X)$.

Remarks

- Since $(x \triangleright y) \triangleright (x \triangleright z) = x \triangleright (y \triangleright z)$ for all $x, y, z \in X$,
$$\varphi_{x \triangleright y} = \varphi_x \varphi_y \varphi_x^{-1}$$

Definition

Let X be a finite rack and $\varphi_x \in Aut(X)$. Let $\varphi_x = \sigma_1\sigma_2...\sigma_k$ be the decomposition of φ_x into product of disjoint cycles σ_i. Let $\ell(\sigma_i) = \ell_i$ is the cycle length of σ_i and $1 \leq \ell_1 \leq \ell_2 \leq ... \leq \ell_k$. Then the list of all ℓ_i is called the **pattern** of φ_x. The **profile** of X is the sequence of patterns of all $\varphi_x \in Aut(X)$.

Remarks

- Since $(x \triangleright y) \triangleright (x \triangleright z) = x \triangleright (y \triangleright z)$ for all $x, y, z \in X$,

$$\varphi_{x \triangleright y} = \varphi_x \varphi_y \varphi_x^{-1}$$

  By using this one can see that, **any two permutations of a finite connected rack X are mutually conjugate**, and hence have same pattern.

Definition

Let X be a finite rack and $\varphi_x \in Aut(X)$. Let $\varphi_x = \sigma_1\sigma_2 ... \sigma_k$ be the decomposition of φ_x into product of disjoint cycles σ_i. Let $\ell(\sigma_i) = \ell_i$ is the cycle length of σ_i and $1 \leq \ell_1 \leq \ell_2 \leq ... \leq \ell_k$. Then the list of all ℓ_i is called the **pattern** of φ_x. The **profile** of X is the sequence of patterns of all $\varphi_x \in Aut(X)$.

Remarks

- Since $(x \triangleright y) \triangleright (x \triangleright z) = x \triangleright (y \triangleright z)$ for all $x, y, z \in X$,

  $$\varphi_{x \triangleright y} = \varphi_x \varphi_y \varphi_x^{-1}$$

  By using this one can see that, **any two permutations of a finite connected rack X are mutually conjugate**, and hence have same pattern. Therefore, the profile of a finite connected rack X is a constant sequence.

Definition

Let X be a finite rack and $\varphi_x \in Aut(X)$. Let $\varphi_x = \sigma_1\sigma_2...\sigma_k$ be the decomposition of φ_x into product of disjoint cycles σ_i. Let $\ell(\sigma_i) = \ell_i$ is the cycle length of σ_i and $1 \leq \ell_1 \leq \ell_2 \leq ... \leq \ell_k$. Then the list of all ℓ_i is called the **pattern** of φ_x. The **profile** of X is the sequence of patterns of all $\varphi_x \in Aut(X)$.

Remarks

- Since $(x \triangleright y) \triangleright (x \triangleright z) = x \triangleright (y \triangleright z)$ for all $x, y, z \in X$,

$$\varphi_{x \triangleright y} = \varphi_x \varphi_y \varphi_x^{-1}$$

  By using this one can see that, **any two permutations of a finite connected rack X are mutually conjugate**, and hence have same pattern. Therefore, the profile of a finite connected rack X is a constant sequence.
- Hayashi called the pattern of any φ_x as the profile of a finite connected rack X for short.

Notation

We write the profile of a finite connected rack X as:

$$Profile(X) = 1^{m_0}\ell_1^{m_1}\ell_2^{m_2}...\ell_k^{m_k},$$

where $1 < \ell_1 < \ell_2 < ... < \ell_k$, and $m_0, m_1, ..., m_k$ are the multiplicities of $1, \ell_1, \ell_2, ..., \ell_k$, respectively.

Notation

We write the profile of a finite connected rack X as:

$$Profile(X) = 1^{m_0} \ell_1^{m_1} \ell_2^{m_2} \ldots \ell_k^{m_k},$$

where $1 < \ell_1 < \ell_2 < \ldots < \ell_k$, and $m_0, m_1, \ldots, m_k$ are the multiplicities of $1, \ell_1, \ell_2, \ldots, \ell_k$, respectively.

Hayashi's Conjecture

Let X be a finite connected quandle with

$$Profile(X) = 1^{m_0} \ell_1^{m_1} \ell_2^{m_2} \ldots \ell_k^{m_k}.$$

Then $\ell_i | \ell_k$ (i.e., ℓ_i divides ℓ_k) for any integer i with $1 \leq i \leq k-1$.

Notation

We write the profile of a finite connected rack X as:

$$Profile(X) = 1^{m_0} \ell_1^{m_1} \ell_2^{m_2} ... \ell_k^{m_k},$$

where $1 < \ell_1 < \ell_2 < ... < \ell_k$, and $m_0, m_1, ..., m_k$ are the multiplicities of $1, \ell_1, \ell_2, ..., \ell_k$, respectively.

Hayashi's Conjecture

Let X be a finite connected quandle with

$$Profile(X) = 1^{m_0} \ell_1^{m_1} \ell_2^{m_2} ... \ell_k^{m_k}.$$

Then $\ell_i | \ell_k$ (i.e., ℓ_i divides ℓ_k) for any integer i with $1 \leq i \leq k - 1$.

Example

$$Profile(SmallQuandle(42, 7)) = 1^2.2^2.3^4.6^4.$$

Observations

Observations

- Hayashi's conjecture is trivially true for dihedral quandle $\mathbb{D}_n$,

Observations

- Hayashi's conjecture is trivially true for dihedral quandle $\mathbb{D}_n$, since for $i, j \in \mathbb{D}_n$, $\varphi_i(j) = 2j - i \pmod{n}$,

Observations

- Hayashi's conjecture is trivially true for dihedral quandle $\mathbb{D}_n$, since for $i, j \in \mathbb{D}_n$, $\varphi_i(j) = 2j - i \pmod{n}$, and
$$\varphi_i = \prod_{j=1}^{[\frac{n-1}{2}]} (i+j \ i-j) \pmod{n}.$$
Hence, $Profile(\mathbb{D}_n) = 1^{m_0} 2^{m_1}$.

Observations

- Hayashi's conjecture is trivially true for dihedral quandle $\mathbb{D}_n$, since for $i, j \in \mathbb{D}_n$, $\varphi_i(j) = 2j - i \pmod{n}$, and
$$\varphi_i = \prod_{j=1}^{[\frac{n-1}{2}]} (i+j\ i-j) \pmod{n}.$$
Hence, $\mathit{Profile}(\mathbb{D}_n) = 1^{m_0} 2^{m_1}$.
- L. Vendramin calculated all connected quandles of size $n \leq 47$. These small quandles are included in a GAP package called **Rig** (Racks in gap) as: **SmallQuandle(n, q(n))**, where $q(n) :=$ quandle number of size n.

Observations

- Hayashi's conjecture is trivially true for dihedral quandle $\mathbb{D}_n$, since for $i, j \in \mathbb{D}_n$, $\varphi_i(j) = 2j - i \pmod{n}$, and

$$\varphi_i = \prod_{j=1}^{[\frac{n-1}{2}]} (i+j \; i-j) \pmod{n}.$$

  Hence, $Profile(\mathbb{D}_n) = 1^{m_0} 2^{m_1}$.

- L. Vendramin calculated all connected quandles of size $n \leq 47$. These small quandles are included in a GAP package called **Rig** (Racks in gap) as: **SmallQuandle(n, q(n))**, where $q(n) :=$ quandle number of size n. Hayashi's conjecture is true for all Rig quandles.

Observations

- Hayashi's conjecture is trivially true for dihedral quandle $\mathbb{D}_n$, since for $i, j \in \mathbb{D}_n$, $\varphi_i(j) = 2j - i \pmod{n}$, and
$$\varphi_i = \prod_{j=1}^{[\frac{n-1}{2}]} (i+j\ i-j) \pmod{n}.$$
Hence, $Profile(\mathbb{D}_n) = 1^{m_0}2^{m_1}$.
- L. Vendramin calculated all connected quandles of size $n \leq 47$. These small quandles are included in a GAP package called **Rig** (Racks in gap) as: **SmallQuandle(n, q(n))**, where $q(n) :=$ quandle number of size n. Hayashi's conjecture is true for all Rig quandles.
- Note that most Rig quandles are affine.

Observations

- Hayashi's conjecture is trivially true for dihedral quandle $\mathbb{D}_n$, since for $i, j \in \mathbb{D}_n$, $\varphi_i(j) = 2j - i \pmod{n}$, and
$$\varphi_i = \prod_{j=1}^{[\frac{n-1}{2}]} (i+j \; i-j) \pmod{n}.$$
Hence, $Profile(\mathbb{D}_n) = 1^{m_0} 2^{m_1}$.
- L. Vendramin calculated all connected quandles of size $n \leq 47$. These small quandles are included in a GAP package called **Rig** (Racks in gap) as: **SmallQuandle(n, q(n))**, where $q(n) :=$ quandle number of size n. Hayashi's conjecture is true for all Rig quandles.
- Note that most Rig quandles are affine. Hayashi's conjecture is true for finite connected affine quandles.

Observations

- Hayashi's conjecture is trivially true for dihedral quandle $\mathbb{D}_n$, since for $i, j \in \mathbb{D}_n$, $\varphi_i(j) = 2j - i \pmod{n}$, and
$$\varphi_i = \prod_{j=1}^{[\frac{n-1}{2}]} (i+j \; i-j) \pmod{n}.$$
Hence, $Profile(\mathbb{D}_n) = 1^{m_0} 2^{m_1}$.
- L. Vendramin calculated all connected quandles of size $n \leq 47$. These small quandles are included in a GAP package called **Rig** (Racks in gap) as: **SmallQuandle(n, q(n))**, where $q(n) :=$ quandle number of size n. Hayashi's conjecture is true for all Rig quandles.
- Note that most Rig quandles are affine. Hayashi's conjecture is true for finite connected affine quandles. How?

Observations

- Hayashi's conjecture is trivially true for dihedral quandle $\mathbb{D}_n$, since for $i, j \in \mathbb{D}_n$, $\varphi_i(j) = 2j - i \pmod{n}$, and
$$\varphi_i = \prod_{j=1}^{[\frac{n-1}{2}]} (i+j \; i-j) \pmod{n}.$$
Hence, $Profile(\mathbb{D}_n) = 1^{m_0} 2^{m_1}$.
- L. Vendramin calculated all connected quandles of size $n \leq 47$. These small quandles are included in a GAP package called **Rig** (Racks in gap) as: **SmallQuandle(n, q(n))**, where $q(n) :=$ quandle number of size n. Hayashi's conjecture is true for all Rig quandles.
- Note that most Rig quandles are affine. Hayashi's conjecture is true for finite connected affine quandles. How? By using Bors's result.

Observations

- Hayashi's conjecture is trivially true for dihedral quandle $\mathbb{D}_n$, since for $i, j \in \mathbb{D}_n$, $\varphi_i(j) = 2j - i \pmod{n}$, and
$$\varphi_i = \prod_{j=1}^{[\frac{n-1}{2}]} (i+j \ i-j) \pmod{n}.$$
Hence, $\mathit{Profile}(\mathbb{D}_n) = 1^{m_0} 2^{m_1}$.
- L. Vendramin calculated all connected quandles of size $n \leq 47$. These small quandles are included in a GAP package called **Rig** (Racks in gap) as: **SmallQuandle(n, q(n))**, where $q(n) :=$ quandle number of size n. Hayashi's conjecture is true for all Rig quandles.
- Note that most Rig quandles are affine. Hayashi's conjecture is true for finite connected affine quandles. How? By using Bors's result.
- Hayashi's conjecture is true for connected quandles of size p, p^2 and p^3.

Observations

- Hayashi's conjecture is trivially true for dihedral quandle $\mathbb{D}_n$, since for $i, j \in \mathbb{D}_n$, $\varphi_i(j) = 2j - i \pmod{n}$, and
$$\varphi_i = \prod_{j=1}^{[\frac{n-1}{2}]} (i+j \; i-j) \pmod{n}.$$
Hence, $\mathit{Profile}(\mathbb{D}_n) = 1^{m_0} 2^{m_1}$.
- L. Vendramin calculated all connected quandles of size $n \leq 47$. These small quandles are included in a GAP package called **Rig** (Racks in gap) as: **SmallQuandle(n, q(n))**, where $q(n) :=$ quandle number of size n. Hayashi's conjecture is true for all Rig quandles.
- Note that most Rig quandles are affine. Hayashi's conjecture is true for finite connected affine quandles. How? By using Bors's result.
- Hayashi's conjecture is true for connected quandles of size p, p^2 and p^3. How?

Observations

- Hayashi's conjecture is trivially true for dihedral quandle $\mathbb{D}_n$, since for $i, j \in \mathbb{D}_n$, $\varphi_i(j) = 2j - i \pmod{n}$, and
$$\varphi_i = \prod_{j=1}^{[\frac{n-1}{2}]} (i+j \ i-j) \pmod{n}.$$
Hence, $Profile(\mathbb{D}_n) = 1^{m_0} 2^{m_1}$.
- L. Vendramin calculated all connected quandles of size $n \leq 47$. These small quandles are included in a GAP package called **Rig** (Racks in gap) as: **SmallQuandle(n, q(n))**, where $q(n) :=$ quandle number of size n. Hayashi's conjecture is true for all Rig quandles.
- Note that most Rig quandles are affine. Hayashi's conjecture is true for finite connected affine quandles. How? By using Bors's result.
- Hayashi's conjecture is true for connected quandles of size p, p^2 and p^3. How? By using Bors's result.

Observations

Observations

- A. Bors proved that: Any automorphism α of a finite nilpotent group has a cycle σ such that $\ell(\sigma) = ord(\alpha)$.

Observations

- A. Bors proved that: Any automorphism α of a finite nilpotent group has a cycle σ such that $\ell(\sigma) = ord(\alpha)$.
- Such a cycle σ is called a **regular cycle or orbit**.

Observations

- A. Bors proved that: Any automorphism α of a finite nilpotent group has a cycle σ such that $\ell(\sigma) = ord(\alpha)$.
- Such a cycle σ is called a **regular cycle or orbit**. For example, $\alpha = (1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9\ 10\ 11)$ has a regular cycle,

Observations

- A. Bors proved that: Any automorphism α of a finite nilpotent group has a cycle σ such that $\ell(\sigma) = ord(\alpha)$.
- Such a cycle σ is called a **regular cycle or orbit**. For example, $\alpha = (1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9\ 10\ 11)$ has a regular cycle, while $\beta = (1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9)$ has no regular cycle.

Observations

- A. Bors proved that: Any automorphism α of a finite nilpotent group has a cycle σ such that $\ell(\sigma) = ord(\alpha)$.
- Such a cycle σ is called a **regular cycle or orbit**. For example, $\alpha = (1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9\ 10\ 11)$ has a regular cycle, while $\beta = (1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9)$ has no regular cycle.
- Recall that $ord(\alpha)$ is the least common multiple (lcm) of the cycle lengths of α.

Observations

- A. Bors proved that: Any automorphism α of a finite nilpotent group has a cycle σ such that $\ell(\sigma) = ord(\alpha)$.
- Such a cycle σ is called a **regular cycle or orbit**. For example, $\alpha = (1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9\ 10\ 11)$ has a regular cycle, while $\beta = (1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9)$ has no regular cycle.
- Recall that $ord(\alpha)$ is the least common multiple (lcm) of the cycle lengths of α. Therefore, if α has a regular cycle σ then all cycle lengths of α divide the $\ell(\sigma) = ord(\alpha)$.

Observations

- A. Bors proved that: Any automorphism α of a finite nilpotent group has a cycle σ such that $\ell(\sigma) = ord(\alpha)$.
- Such a cycle σ is called a **regular cycle or orbit**. For example, $\alpha = (1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9\ 10\ 11)$ has a regular cycle, while $\beta = (1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9)$ has no regular cycle.
- Recall that $ord(\alpha)$ is the least common multiple (lcm) of the cycle lengths of α. Therefore, if α has a regular cycle σ then all cycle lengths of α divide the $\ell(\sigma) = ord(\alpha)$.
- For an affine quandle $Aff(A, \alpha)$ we have:

$$\varphi_x(y) = x \triangleright y = (1 - \alpha)(x) + \alpha(y).$$

Observations

- A. Bors proved that: Any automorphism α of a finite nilpotent group has a cycle σ such that $\ell(\sigma) = ord(\alpha)$.
- Such a cycle σ is called a **regular cycle or orbit**. For example, $\alpha = (1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9\ 10\ 11)$ has a regular cycle, while $\beta = (1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9)$ has no regular cycle.
- Recall that $ord(\alpha)$ is the least common multiple (lcm) of the cycle lengths of α. Therefore, if α has a regular cycle σ then all cycle lengths of α divide the $\ell(\sigma) = ord(\alpha)$.
- For an affine quandle $Aff(A, \alpha)$ we have:

$$\varphi_x(y) = x \triangleright y = (1 - \alpha)(x) + \alpha(y).$$

If we take $x = e$, the identity of A, then $\varphi_e(y) = \alpha(y)$.

Observations

- A. Bors proved that: Any automorphism α of a finite nilpotent group has a cycle σ such that $\ell(\sigma) = ord(\alpha)$.
- Such a cycle σ is called a **regular cycle or orbit**. For example, $\alpha = (1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9\ 10\ 11)$ has a regular cycle, while $\beta = (1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9)$ has no regular cycle.
- Recall that $ord(\alpha)$ is the least common multiple (lcm) of the cycle lengths of α. Therefore, if α has a regular cycle σ then all cycle lengths of α divide the $\ell(\sigma) = ord(\alpha)$.
- For an affine quandle $Aff(A, \alpha)$ we have:

  $$\varphi_x(y) = x \triangleright y = (1 - \alpha)(x) + \alpha(y).$$

  If we take $x = e$, the identity of A, then $\varphi_e(y) = \alpha(y)$. Hence, the cycle structure of φ_e is same as the cycle structure of $\alpha \in Aut(A)$.

Observations

- A. Bors proved that: Any automorphism α of a finite nilpotent group has a cycle σ such that $\ell(\sigma) = ord(\alpha)$.
- Such a cycle σ is called a **regular cycle or orbit**. For example, $\alpha = (1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9\ 10\ 11)$ has a regular cycle, while $\beta = (1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9)$ has no regular cycle.
- Recall that $ord(\alpha)$ is the least common multiple (lcm) of the cycle lengths of α. Therefore, if α has a regular cycle σ then all cycle lengths of α divide the $\ell(\sigma) = ord(\alpha)$.
- For an affine quandle $Aff(A, \alpha)$ we have:

$$\varphi_x(y) = x \triangleright y = (1 - \alpha)(x) + \alpha(y).$$

  If we take $x = e$, the identity of A, then $\varphi_e(y) = \alpha(y)$. Hence, the cycle structure of φ_e is same as the cycle structure of $\alpha \in Aut(A)$. Now, since an abelian group is nilpotent, the automorphism α has a regular cycle.

Observations

- Connected quandles of size p and p^2 are affine [by P. Etingof, R. Guralnik, A. Soloviev (2001), and M. Graña (2004)].

Observations

- Connected quandles of size p and p^2 are affine [by P. Etingof, R. Guralnik, A. Soloviev (2001), and M. Graña (2004)]. Therefore, Hayashi's conjecture is true for connected quandles of size p and p^2.

Observations

- Connected quandles of size p and p^2 are affine [by P. Etingof, R. Guralnik, A. Soloviev (2001), and M. Graña (2004)]. Therefore, Hayashi's conjecture is true for connected quandles of size p and p^2.
- A connected quandle of size p^3 is either affine or isomorphic to a coset quandle (G, H, α), where the group G has order p^4 [by G. Bianco's PhD thesis (2015)].

Observations

- Connected quandles of size p and p^2 are affine [by P. Etingof, R. Guralnik, A. Soloviev (2001), and M. Graña (2004)]. Therefore, Hayashi's conjecture is true for connected quandles of size p and p^2.
- A connected quandle of size p^3 is either affine or isomorphic to a coset quandle (G, H, α), where the group G has order p^4 [by G. Bianco's PhD thesis (2015)]. Now, recall that for the coset quandle (G, H, α):

$$gH \triangleright fH = g\alpha(g^{-1}f)H.$$

Therefore, $\varphi_H(fH) = \alpha(f)H$.

Observations

- Connected quandles of size p and p^2 are affine [by P. Etingof, R. Guralnik, A. Soloviev (2001), and M. Graña (2004)]. Therefore, Hayashi's conjecture is true for connected quandles of size p and p^2.
- A connected quandle of size p^3 is either affine or isomorphic to a coset quandle (G, H, α), where the group G has order p^4 [by G. Bianco's PhD thesis (2015)]. Now, recall that for the coset quandle (G, H, α):

  $$gH \triangleright fH = g\alpha(g^{-1}f)H.$$

  Therefore, $\varphi_H(fH) = \alpha(f)H$. Since a group of prime power order is nilpotent, Hayashi's conjecture is true for all connected quandles of size p^3.

Observations

- Connected quandles of size p and p^2 are affine [by P. Etingof, R. Guralnik, A. Soloviev (2001), and M. Graña (2004)]. Therefore, Hayashi's conjecture is true for connected quandles of size p and p^2.
- A connected quandle of size p^3 is either affine or isomorphic to a coset quandle (G, H, α), where the group G has order p^4 [by G. Bianco's PhD thesis (2015)]. Now, recall that for the coset quandle (G, H, α):

  $$gH \triangleright fH = g\alpha(g^{-1}f)H.$$

  Therefore, $\varphi_H(fH) = \alpha(f)H$. Since a group of prime power order is nilpotent, Hayashi's conjecture is true for all connected quandles of size p^3.
- The results of A. Bors, S. Guest and P. Spiga can be used for case-by-case analysis of some other known families of connected quandles.

Subracks of a Connected Rack

Subracks of a Connected Rack

Let X be a finite connected rack. For any subset Y of X, the subrack of X generated by Y is the smallest subrack of X containing Y. For any subrack $Y \subseteq X$ let $Y^c = X \setminus Y$.

Subracks of a Connected Rack

Let X be a finite connected rack. For any subset Y of X, the subrack of X generated by Y is the smallest subrack of X containing Y. For any subrack $Y \subseteq X$ let $Y^c = X \setminus Y$.

Lemma 1

Let Y be a subrack of X with $Y \neq X$. Then X is generated by Y^c.

Subracks of a Connected Rack

Let X be a finite connected rack. For any subset Y of X, the subrack of X generated by Y is the smallest subrack of X containing Y. For any subrack $Y \subseteq X$ let $Y^c = X \setminus Y$.

Lemma 1

Let Y be a subrack of X with $Y \neq X$. Then X is generated by Y^c.

Proof.

Since Y is a subrack of X, we conclude that $Y \triangleright Y^c = Y^c$. Let

$$Z = \{y_1 \triangleright (y_2 \triangleright \dots \triangleright (y_{n-1} \triangleright y_n)) \mid n \geq 1, y_1, \dots, y_n \in Y^c\}.$$

Then $y \triangleright z \in Z$ for all $y \in Y^c, z \in Z$ by definition, and $y \triangleright z \in Z$ for all $y \in Y$ by the self-distributivity of $\triangleright$ and the Y-invariance of Y^c. Hence Z is a non-empty X-invariant subset of X, and therefore equal to X since X is connected.

Subracks of a Connected Rack

Lemma 2

Let X be a connected rack such that $X = Y \cup Z$, for two subracks Y and Z of X. Then $X = Y$ or $X = Z$.

Subracks of a Connected Rack

Lemma 2

Let X be a connected rack such that $X = Y \cup Z$, for two subracks Y and Z of X. Then $X = Y$ or $X = Z$.

Proof.

Assume that $X \neq Y$, then X is generated by $Y^c \subseteq Z$, by Lemma 1. Since Z is a subrack of X, one concludes that $X = Z$.

Lemma 2

Let X be a connected rack such that $X = Y \cup Z$, for two subracks Y and Z of X. Then $X = Y$ or $X = Z$.

Proof.

Assume that $X \neq Y$, then X is generated by $Y^c \subseteq Z$, by Lemma 1. Since Z is a subrack of X, one concludes that $X = Z$.

Corollary 1

Let X be a connected rack and $x \in X$. Let $p, q \in \mathbb{N}_{\geq 2}$, and $Y = \{y \in X \mid \varphi_x^p(y) = y\}, Z = \{z \in X \mid \varphi_x^q(z) = z\}$. Assume that $X = Y \cup Z$. Then $X = Y$ or $X = Z$.

Obstruction on the Profile of a Connected Rack

Obstruction on the Profile of a Connected Rack

Proposition 1.

There is no finite connected rack X (respectively, quandle) of profile $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}...\ell_k^{m_k}$ such that $lcm(\ell_1, \ell_2, ..., \ell_i)$ and $lcm(\ell_{i+1}, \ell_{i+2}, ..., \ell_k)$ do not divide each other.

Obstruction on the Profile of a Connected Rack

Proposition 1.

There is no finite connected rack X (respectively, quandle) of profile $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}...\ell_k^{m_k}$ such that $lcm(\ell_1, \ell_2, ..., \ell_i)$ and $lcm(\ell_{i+1}, \ell_{i+2}, ..., \ell_k)$ do not divide each other.

Proof.

Suppose that there exists a finite connected rack X with given profile. Let $p = lcm(\ell_1, \ell_2, ..., \ell_i)$, and $q = lcm(\ell_{i+1}, \ell_{i+2}, ..., \ell_k)$. Then,

$$Y = \{y \in X \mid \varphi_x^p(y) = y\}, Z = \{z \in X \mid \varphi_x^q(z) = z\}.$$

By the self-distributivity of $\triangleright$, the sets Y and Z are subracks of X. Then $X = Y \cup Z$ by definition of p and q and, $X \neq Y$ and $X \neq Z$, a contradiction to Corollary 1.

Obstruction on the Profile of a Connected Rack

Obstruction on the Profile of a Connected Rack

- By above results, there is no finite connected rack X of profile $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}$ with $\ell_1 \nmid \ell_2$.

Obstruction on the Profile of a Connected Rack

- By above results, there is no finite connected rack X of profile $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}$ with $\ell_1 \nmid \ell_2$. Therefore, Hayashi's conjecture is true for all finite connected racks and quandles with profiles $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}$.

Obstruction on the Profile of a Connected Rack

- By above results, there is no finite connected rack X of profile $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}$ with $\ell_1 \nmid \ell_2$. Therefore, Hayashi's conjecture is true for all finite connected racks and quandles with profiles $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}$.
- For the profile $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}\ell_3^{m_3}$, we have the following possibilities.

Obstruction on the Profile of a Connected Rack

- By above results, there is no finite connected rack X of profile $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}$ with $\ell_1 \nmid \ell_2$. Therefore, Hayashi's conjecture is true for all finite connected racks and quandles with profiles $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}$.
- For the profile $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}\ell_3^{m_3}$, we have the following possibilities.
  - If $\ell_1, \ell_2 \mid \ell_3$ then Hayashi's conjecture is true.

Obstruction on the Profile of a Connected Rack

- By above results, there is no finite connected rack X of profile $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}$ with $\ell_1 \nmid \ell_2$. Therefore, Hayashi's conjecture is true for all finite connected racks and quandles with profiles $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}$.
- For the profile $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}\ell_3^{m_3}$, we have the following possibilities.
  - If $\ell_1, \ell_2 \mid \ell_3$ then Hayashi's conjecture is true.
  - If $\ell_i \nmid \ell_3$ or $\ell_i \nmid \ell_3$ and $\ell_j \mid \ell_3$ for distinct i, j in $\{1, 2\}$, then we have further two cases to consider.

Obstruction on the Profile of a Connected Rack

- By above results, there is no finite connected rack X of profile $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}$ with $\ell_1 \nmid \ell_2$. Therefore, Hayashi's conjecture is true for all finite connected racks and quandles with profiles $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}$.
- For the profile $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}\ell_3^{m_3}$, we have the following possibilities.
  - If $\ell_1, \ell_2 \mid \ell_3$ then Hayashi's conjecture is true.
  - If $\ell_i \nmid \ell_3$ or $\ell_i \nmid \ell_3$ and $\ell_j \mid \ell_3$ for distinct i, j in $\{1, 2\}$, then we have further two cases to consider.

  - The case when $\ell_k \nmid lcm(\ell_{k+1}, \ell_{k+2}) \pmod 3$ for $k \in \{1, 2, 3\}$, is excluded by Proposition 1.

Obstruction on the Profile of a Connected Rack

- By above results, there is no finite connected rack X of profile $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}$ with $\ell_1 \nmid \ell_2$. Therefore, Hayashi's conjecture is true for all finite connected racks and quandles with profiles $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}$.
- For the profile $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}\ell_3^{m_3}$, we have the following possibilities.
  - If $\ell_1, \ell_2 \mid \ell_3$ then Hayashi's conjecture is true.
  - If $\ell_i \nmid \ell_3$ or $\ell_i \nmid \ell_3$ and $\ell_j \mid \ell_3$ for distinct i, j in $\{1, 2\}$, then we have further two cases to consider.

  - The case when $\ell_k \nmid lcm(\ell_{k+1}, \ell_{k+2}) \pmod 3$ for $k \in \{1, 2, 3\}$, is excluded by Proposition 1.
  - The case when $\ell_k \mid lcm(\ell_{k+1}, \ell_{k+2}) \pmod 3$ for $k \in \{1, 2, 3\}$, can not be excluded by Proposition 1.

Obstruction on the Profile of a Connected Rack

- By above results, there is no finite connected rack X of profile $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}$ with $\ell_1 \nmid \ell_2$. Therefore, Hayashi's conjecture is true for all finite connected racks and quandles with profiles $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}$.
- For the profile $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}\ell_3^{m_3}$, we have the following possibilities.
  - If $\ell_1, \ell_2 \mid \ell_3$ then Hayashi's conjecture is true.
  - If $\ell_i \nmid \ell_3$ or $\ell_i \nmid \ell_3$ and $\ell_j \mid \ell_3$ for distinct i, j in $\{1, 2\}$, then we have further two cases to consider.

  - The case when $\ell_k \nmid lcm(\ell_{k+1}, \ell_{k+2}) \pmod 3$ for $k \in \{1, 2, 3\}$, is excluded by Proposition 1.
  - The case when $\ell_k \mid lcm(\ell_{k+1}, \ell_{k+2}) \pmod 3$ for $k \in \{1, 2, 3\}$, can not be excluded by Proposition 1. One such profile is:

Obstruction on the Profile of a Connected Rack

- By above results, there is no finite connected rack X of profile $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}$ with $\ell_1 \nmid \ell_2$. Therefore, Hayashi's conjecture is true for all finite connected racks and quandles with profiles $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}$.
- For the profile $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}\ell_3^{m_3}$, we have the following possibilities.
  - If $\ell_1, \ell_2 \mid \ell_3$ then Hayashi's conjecture is true.
  - If $\ell_i \nmid \ell_3$ or $\ell_i \nmid \ell_3$ and $\ell_j \mid \ell_3$ for distinct i, j in $\{1, 2\}$, then we have further two cases to consider.

  - The case when $\ell_k \nmid lcm(\ell_{k+1}, \ell_{k+2}) \pmod 3$ for $k \in \{1, 2, 3\}$, is excluded by Proposition 1.
  - The case when $\ell_k \mid lcm(\ell_{k+1}, \ell_{k+2}) \pmod 3$ for $k \in \{1, 2, 3\}$, can not be excluded by Proposition 1. One such profile is: $1^{m_0}\ell_1\ell_2\ell_3$ with $(\ell_1, \ell_2, \ell_3) = (pq, pr, qr)$ for pairwise distinct primes p, q, r.

Obstruction on the Profile of a Connected Rack

- By above results, there is no finite connected rack X of profile $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}$ with $\ell_1 \nmid \ell_2$. Therefore, Hayashi's conjecture is true for all finite connected racks and quandles with profiles $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}$.
- For the profile $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}\ell_3^{m_3}$, we have the following possibilities.
  - If $\ell_1, \ell_2 \mid \ell_3$ then Hayashi's conjecture is true.
  - If $\ell_i \nmid \ell_3$ or $\ell_i \nmid \ell_3$ and $\ell_j \mid \ell_3$ for distinct i, j in $\{1, 2\}$, then we have further two cases to consider.

  - The case when $\ell_k \nmid lcm(\ell_{k+1}, \ell_{k+2}) \pmod 3$ for $k \in \{1, 2, 3\}$, is excluded by Proposition 1.
  - The case when $\ell_k \mid lcm(\ell_{k+1}, \ell_{k+2}) \pmod 3$ for $k \in \{1, 2, 3\}$, can not be excluded by Proposition 1. One such profile is: $1^{m_0}\ell_1\ell_2\ell_3$ with $(\ell_1, \ell_2, \ell_3) = (pq, pr, qr)$ for pairwise distinct primes p, q, r. For example:

$$1^{m_0}6.10.15 = 1^{m_0}(2.3)(2.5)(3.5) \text{ with } (p, q, r) = (2, 3, 5).$$

Obstruction on the Profile of a Connected Rack

Obstruction on the Profile of a Connected Rack

Proposition 2

There is no finite connected crossed set with profile $1^{m_0}\ell_1\ell_2\ell_3$, where $(\ell_1, \ell_2, \ell_3) = (pq, pr, qr)$ for pairwise distinct primes p, q, r.

Obstruction on the Profile of a Connected Rack

Proposition 2

There is no finite connected crossed set with profile $1^{m_0}\ell_1\ell_2\ell_3$, where $(\ell_1, \ell_2, \ell_3) = (pq, pr, qr)$ for pairwise distinct primes p, q, r.

Sketch of Proof

Obstruction on the Profile of a Connected Rack

Proposition 2

There is no finite connected crossed set with profile $1^{m_0}\ell_1\ell_2\ell_3$, where $(\ell_1, \ell_2, \ell_3) = (pq, pr, qr)$ for pairwise distinct primes p, q, r.

Sketch of Proof

Suppose that there exists a finite connected crossed set X with given profile.

Proposition 2

There is no finite connected crossed set with profile $1^{m_0}\ell_1\ell_2\ell_3$, where $(\ell_1, \ell_2, \ell_3) = (pq, pr, qr)$ for pairwise distinct primes p, q, r.

Sketch of Proof

Suppose that there exists a finite connected crossed set X with given profile. Let $x \in X$, $t \geq 1$,

$$X_t = \{y \in X \mid \varphi_x^t(y) = y\},$$

and $X'_t = X_t \setminus X_1$ for all $t > 1$.

Obstruction on the Profile of a Connected Rack

Proposition 2

There is no finite connected crossed set with profile $1^{m_0}\ell_1\ell_2\ell_3$, where $(\ell_1, \ell_2, \ell_3) = (pq, pr, qr)$ for pairwise distinct primes p, q, r.

Sketch of Proof

Suppose that there exists a finite connected crossed set X with given profile. Let $x \in X$, $t \geq 1$,

$$X_t = \{y \in X \mid \varphi_x^t(y) = y\},$$

and $X'_t = X_t \setminus X_1$ for all $t > 1$. Then X is the disjoint union of non-empty sets $X_1, X'_{pq}, X'_{pr}, X'_{qr}$.

Proposition 2

There is no finite connected crossed set with profile $1^{m_0}\ell_1\ell_2\ell_3$, where $(\ell_1, \ell_2, \ell_3) = (pq, pr, qr)$ for pairwise distinct primes p, q, r.

Sketch of Proof

Suppose that there exists a finite connected crossed set X with given profile. Let $x \in X$, $t \geq 1$,

$$X_t = \{y \in X \mid \varphi_x^t(y) = y\},$$

and $X_t' = X_t \setminus X_1$ for all $t > 1$. Then X is the disjoint union of non-empty sets $X_1, X_{pq}', X_{pr}', X_{qr}'$. Now we have the following steps.

Step 1

X_t is a non-empty subrack of X. In particular,

$$y \triangleright (X \setminus X_t) = X \setminus X_t \text{ for all } y \in X_t.$$

Step 1

X_t is a non-empty subrack of X. In particular,

$$y \triangleright (X \setminus X_t) = X \setminus X_t \text{ for all } y \in X_t.$$

Step 2

For all $y \in X'_{pq}$, there exists $z \in X'_{pr}$, such that $y \triangleright z \neq z$.

Step 1

X_t is a non-empty subrack of X. In particular,

$$y \triangleright (X \setminus X_t) = X \setminus X_t \text{ for all } y \in X_t.$$

Step 2

For all $y \in X'_{pq}$, there exists $z \in X'_{pr}$, such that $y \triangleright z \neq z$.

Step 3

Let $y \in X'_{pq}$ and $z \in X \setminus X_{pq}$ with $y \triangleright z \neq z$. Let t be the smallest positive integer with $\varphi_y^t(z) = z$, then $t = pr$ or $t = qr$.

Step 1

X_t is a non-empty subrack of X. In particular,

$$y \triangleright (X \setminus X_t) = X \setminus X_t \text{ for all } y \in X_t.$$

Step 2

For all $y \in X'_{pq}$, there exists $z \in X'_{pr}$, such that $y \triangleright z \neq z$.

Step 3

Let $y \in X'_{pq}$ and $z \in X \setminus X_{pq}$ with $y \triangleright z \neq z$. Let t be the smallest positive integer with $\varphi_y^t(z) = z$, then $t = pr$ or $t = qr$.

Step 4

Let $y \in X'_{pq}$. Then there exist $z \in X'_{pr}, f \in X'_{qr}$ such that $z \triangleright f = y$ or $f \triangleright z = y$.

Step 5

Assume that X'_{pr} and X'_{qr} are subracks of X. Then X'_{pq} is not a subrack of X.

Step 5

Assume that X'_{pr} and X'_{qr} are subracks of X. Then X'_{pq} is not a subrack of X.

Step 6

Let $y \in X'_{pq}$. Then there exist $z \in X'_{pr}$ and $f \in X'_{qr}$ with $y \triangleright z \in X'_{qr}$, $y \triangleright f \in X'_{pr}$.

Step 5

Assume that X'_{pr} and X'_{qr} are subracks of X. Then X'_{pq} is not a subrack of X.

Step 6

Let $y \in X'_{pq}$. Then there exist $z \in X'_{pr}$ and $f \in X'_{qr}$ with $y \triangleright z \in X'_{qr}$, $y \triangleright f \in X'_{pr}$.

Step 7

Let $y \in X'_{pq}$. Then $y \triangleright x \in X'_{pq}$.

Step 5

Assume that X'_{pr} and X'_{qr} are subracks of X. Then X'_{pq} is not a subrack of X.

Step 6

Let $y \in X'_{pq}$. Then there exist $z \in X'_{pr}$ and $f \in X'_{qr}$ with $y \triangleright z \in X'_{qr}$, $y \triangleright f \in X'_{pr}$.

Step 7

Let $y \in X'_{pq}$. Then $y \triangleright x \in X'_{pq}$.

Step 8

Let $y \in X'_{pq}$. Let $\varphi_y = \sigma_1\sigma_2\sigma_3$ be the decomposition of φ_y into the product of a $pq-$, $pr-$, and $qr-$cycle. Then $supp(\sigma_1) \subseteq X_{pq}$ and $supp(\sigma_2), supp(\sigma_3) \subseteq X'_{pr} \cup X'_{qr}$.

Sketch of Proof Continue

Let $y \in X'_{pq}$ and $z \in X'_{pr}$, then $z \rhd x \neq x$ and $\varphi_z^{pr}(x) = x$, by Step 7. Therefore $\varphi^{pr}_{y \rhd z}(y \rhd x) = y \rhd x$. Moreover, $y \rhd x \in X'_{pq}$, by Step 7. Step 1 implies that $y \rhd z \in X'_{pr} \cup X'_{qr}$. If $y \rhd z \in X'_{pr}$, then the entries of $pr-$cycle of $\varphi_{y \rhd z}$ belong to X_{pr}, by Step 8, in contradiction to $y \rhd x \in X'_{pq}$. Thus $y \rhd z \in X'_{qr}$, which implies that $y \rhd X'_{pr} \subseteq X'_{qr}$ and $y \rhd X'_{qr} \subseteq X'_{pr}$ by symmetry. This is impossible since $\mid X'_{pr} \mid = pr \neq qr = \mid X'_{qr} \mid$.

Sketch of Proof Continue

Let $y \in X'_{pq}$ and $z \in X'_{pr}$, then $z \triangleright x \neq x$ and $\varphi_z^{pr}(x) = x$, by Step 7. Therefore $\varphi_{y \triangleright z}^{pr}(y \triangleright x) = y \triangleright x$. Moreover, $y \triangleright x \in X'_{pq}$, by Step 7. Step 1 implies that $y \triangleright z \in X'_{pr} \cup X'_{qr}$. If $y \triangleright z \in X'_{pr}$, then the entries of $pr-$cycle of $\varphi_{y \triangleright z}$ belong to X_{pr}, by Step 8, in contradiction to $y \triangleright x \in X'_{pq}$. Thus $y \triangleright z \in X'_{qr}$, which implies that $y \triangleright X'_{pr} \subseteq X'_{qr}$ and $y \triangleright X'_{qr} \subseteq X'_{pr}$ by symmetry. This is impossible since $\mid X'_{pr} \mid = pr \neq qr = \mid X'_{qr} \mid$.

Remarks

Sketch of Proof Continue

Let $y \in X'_{pq}$ and $z \in X'_{pr}$, then $z \triangleright x \neq x$ and $\varphi_z^{pr}(x) = x$, by Step 7. Therefore $\varphi_{y \triangleright z}^{pr}(y \triangleright x) = y \triangleright x$. Moreover, $y \triangleright x \in X'_{pq}$, by Step 7. Step 1 implies that $y \triangleright z \in X'_{pr} \cup X'_{qr}$. If $y \triangleright z \in X'_{pr}$, then the entries of $pr-$cycle of $\varphi_{y \triangleright z}$ belong to X_{pr}, by Step 8, in contradiction to $y \triangleright x \in X'_{pq}$. Thus $y \triangleright z \in X'_{qr}$, which implies that $y \triangleright X'_{pr} \subseteq X'_{qr}$ and $y \triangleright X'_{qr} \subseteq X'_{pr}$ by symmetry. This is impossible since $\mid X'_{pr} \mid = pr \neq qr = \mid X'_{qr} \mid$.

Remarks

- By Proposition 2, there is no connected crossed set with profile $1^{m_0}6.10.15$.

Sketch of Proof Continue

Let $y \in X'_{pq}$ and $z \in X'_{pr}$, then $z \triangleright x \neq x$ and $\varphi_z^{pr}(x) = x$, by Step 7. Therefore $\varphi_{y \triangleright z}^{pr}(y \triangleright x) = y \triangleright x$. Moreover, $y \triangleright x \in X'_{pq}$, by Step 7. Step 1 implies that $y \triangleright z \in X'_{pr} \cup X'_{qr}$. If $y \triangleright z \in X'_{pr}$, then the entries of $pr-$cycle of $\varphi_{y \triangleright z}$ belong to X_{pr}, by Step 8, in contradiction to $y \triangleright x \in X'_{pq}$. Thus $y \triangleright z \in X'_{qr}$, which implies that $y \triangleright X'_{pr} \subseteq X'_{qr}$ and $y \triangleright X'_{qr} \subseteq X'_{pr}$ by symmetry. This is impossible since $\mid X'_{pr} \mid = pr \neq qr = \mid X'_{qr} \mid$.

Remarks

- By Proposition 2, there is no connected crossed set with profile $1^{m_0}6.10.15$.
- By Propositions 1, 2, and Corollary 1, Hayashi's conjecture is true for any connected crossed set X with $\varphi_x \in Aut(X)$ such that $supp(\varphi_x) \leq 31$.

Sketch of Proof Continue

Let $y \in X'_{pq}$ and $z \in X'_{pr}$, then $z \triangleright x \neq x$ and $\varphi_z^{pr}(x) = x$, by Step 7. Therefore $\varphi_{y \triangleright z}^{pr}(y \triangleright x) = y \triangleright x$. Moreover, $y \triangleright x \in X'_{pq}$, by Step 7. Step 1 implies that $y \triangleright z \in X'_{pr} \cup X'_{qr}$. If $y \triangleright z \in X'_{pr}$, then the entries of $pr-$cycle of $\varphi_{y \triangleright z}$ belong to X_{pr}, by Step 8, in contradiction to $y \triangleright x \in X'_{pq}$. Thus $y \triangleright z \in X'_{qr}$, which implies that $y \triangleright X'_{pr} \subseteq X'_{qr}$ and $y \triangleright X'_{qr} \subseteq X'_{pr}$ by symmetry. This is impossible since $\mid X'_{pr} \mid = pr \neq qr = \mid X'_{qr} \mid$.

Remarks

- By Proposition 2, there is no connected crossed set with profile $1^{m_0}6.10.15$.
- By Propositions 1, 2, and Corollary 1, Hayashi's conjecture is true for any connected crossed set X with $\varphi_x \in Aut(X)$ such that $supp(\varphi_x) \leq 31$.
- Proposition 2 is a particular case of the following theorem.

Obstruction on the Profile of a Connected Rack

Theorem

Let $p_1, p_2, ..., p_r$ be pairwise distinct primes for positive integer r. Let

$$\ell_1 = \prod_{i=1}^{r} p_i^{a_i},\ \ell_2 = \prod_{i=1}^{r} p_i^{b_i} \text{ and } \ell_3 = \prod_{i=1}^{r} p_i^{c_i},$$

for non-negative integers a_i, b_i and c_i for all $1 \leq i \leq r$. Let $1 < \ell_1 < \ell_2 < \ell_3$, $\ell_1 \nmid \ell_3$, $\ell_2 \nmid \ell_3$ and $\ell_k \mid lcm(\ell_{k+1}, \ell_{k+2}) \pmod 3$ for $k \in \{1, 2, 3\}$. Then there is no finite connected crossed set X with profile $1^{m_0}\ell_1\ell_2\ell_3$.

Obstruction on the Profile of a Connected Rack

Theorem

Let $p_1, p_2, ..., p_r$ *be pairwise distinct primes for positive integer* r. *Let*

$$\ell_1 = \prod_{i=1}^{r} p_i^{a_i},\ \ell_2 = \prod_{i=1}^{r} p_i^{b_i} \text{ and } \ell_3 = \prod_{i=1}^{r} p_i^{c_i},$$

for non-negative integers a_i, b_i *and* c_i *for all* $1 \leq i \leq r$. *Let* $1 < \ell_1 < \ell_2 < \ell_3$, $\ell_1 \nmid \ell_3$, $\ell_2 \nmid \ell_3$ *and* $\ell_k \mid lcm(\ell_{k+1}, \ell_{k+2}) \pmod 3$ *for* $k \in \{1, 2, 3\}$. *Then there is no finite connected crossed set* X *with profile* $1^{m_0}\ell_1\ell_2\ell_3$.

Obstruction on the Profile of a Connected Rack

Theorem

Let $p_1, p_2, ..., p_r$ be pairwise distinct primes for positive integer r. Let

$$\ell_1 = \prod_{i=1}^{r} p_i^{a_i},\ \ell_2 = \prod_{i=1}^{r} p_i^{b_i} \text{ and } \ell_3 = \prod_{i=1}^{r} p_i^{c_i},$$

for non-negative integers a_i, b_i and c_i for all $1 \leq i \leq r$. Let $1 < \ell_1 < \ell_2 < \ell_3$, $\ell_1 \nmid \ell_3$, $\ell_2 \nmid \ell_3$ and $\ell_k \mid lcm(\ell_{k+1}, \ell_{k+2}) \pmod 3$ for $k \in \{1, 2, 3\}$. Then there is no finite connected crossed set X with profile $1^{m_0}\ell_1\ell_2\ell_3$.

Questions, comments, suggestions???