Permutation groups, permutation patterns, and Galois connections

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Institute of Algebra

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Workshop on General Algebra
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Outline

Permutation patterns

The "Galois" connections $\mathrm{Pat}^{(\ell)} - \mathrm{Comp}^{(n)}$ and $\mathrm{gPat}^{(\ell)} - \mathrm{gComp}^{(n)}$

The Galois closures $\mathrm{gComp}^{(n)}\, G$ and Galois kernels $\mathrm{gPat}^{(\ell)}\, H$

Remarks and references

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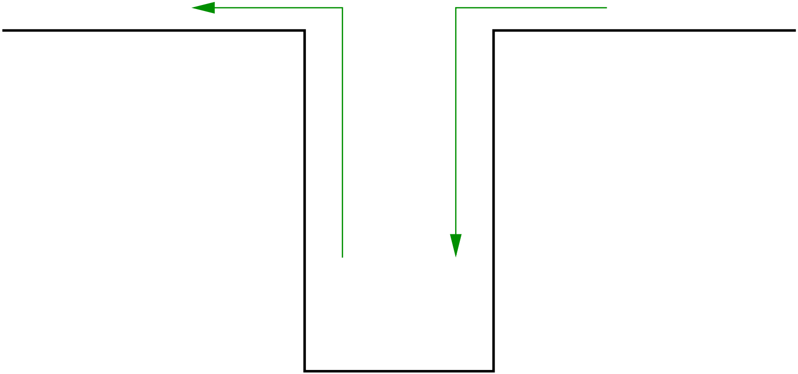
Remarks and references

Some motivation

Which number sequences (permutations) can be sorted by a stack?



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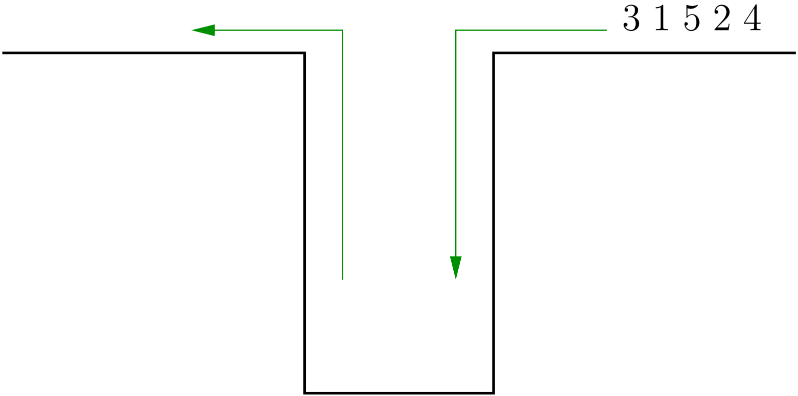

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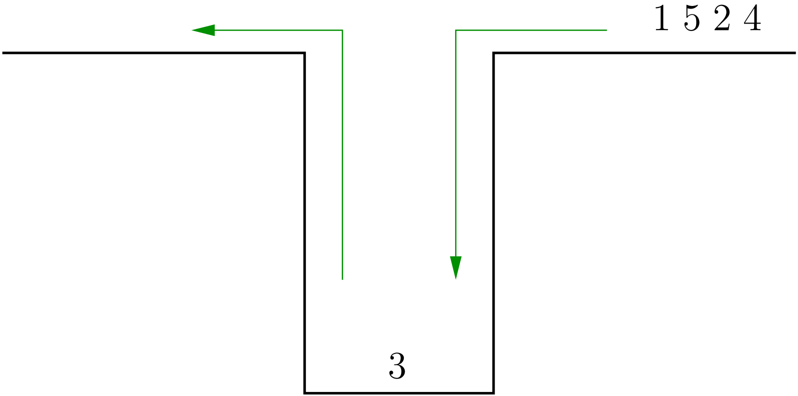

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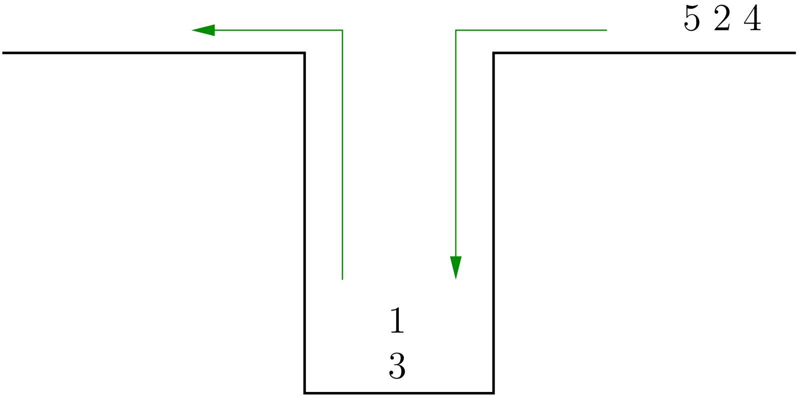

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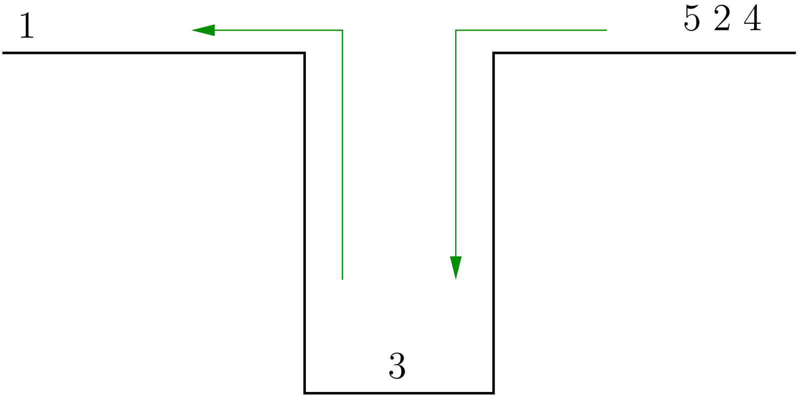

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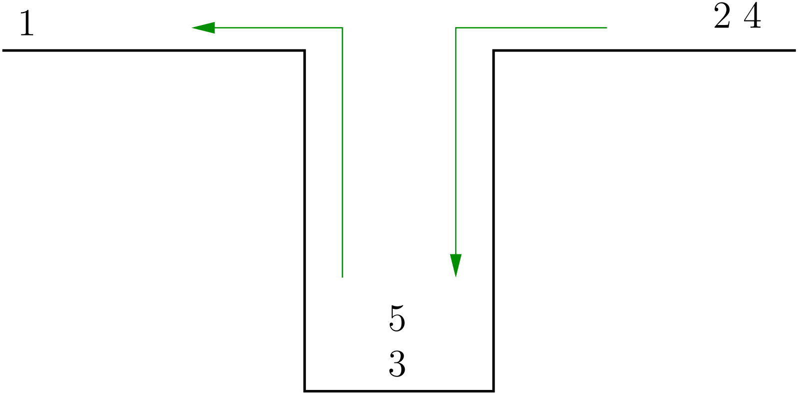

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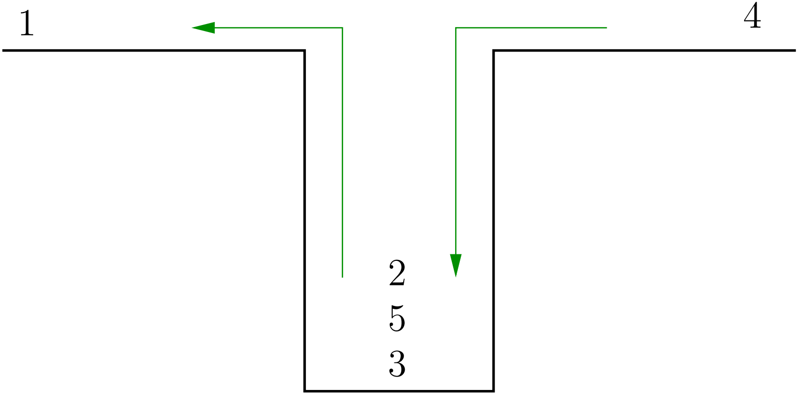

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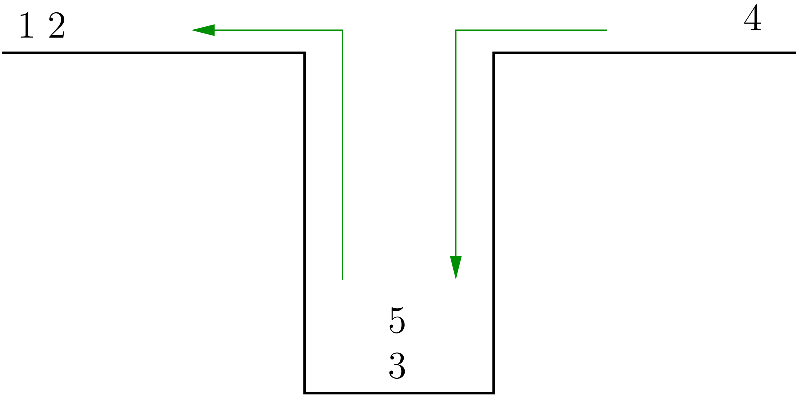

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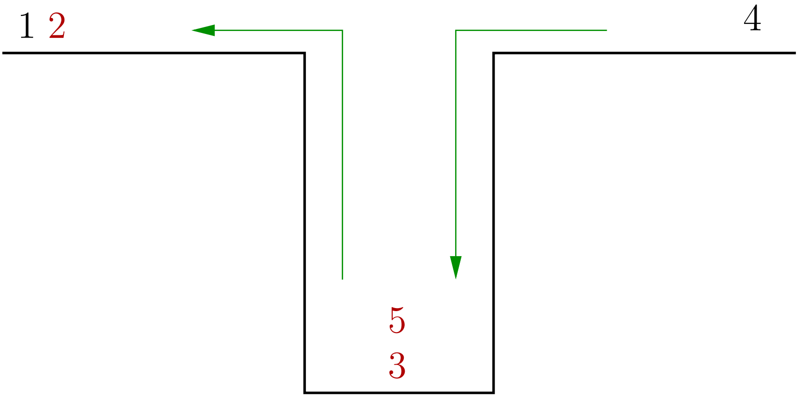

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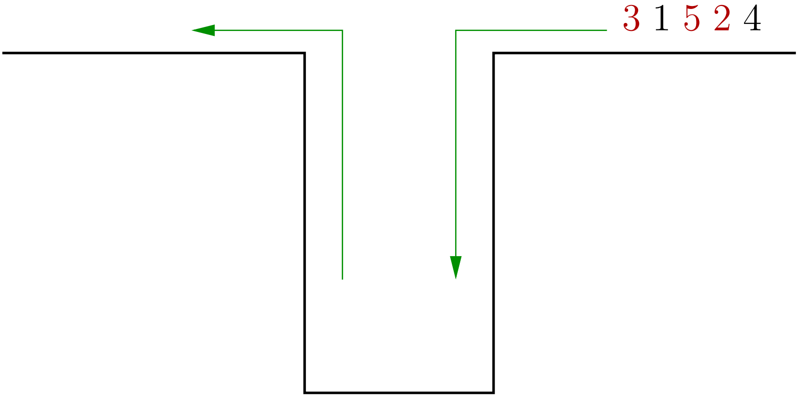

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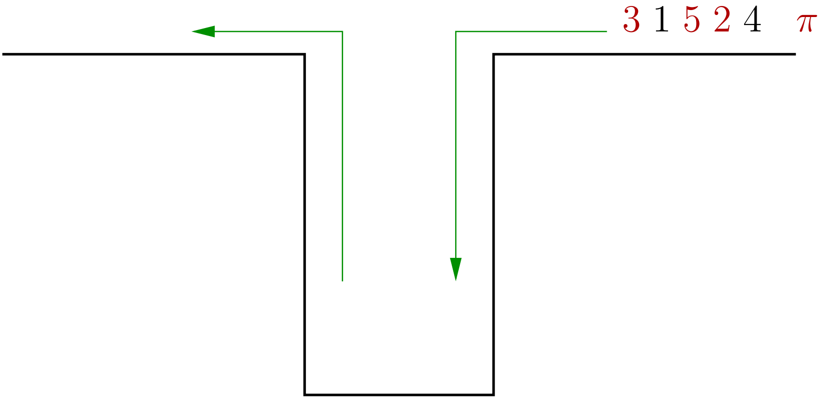

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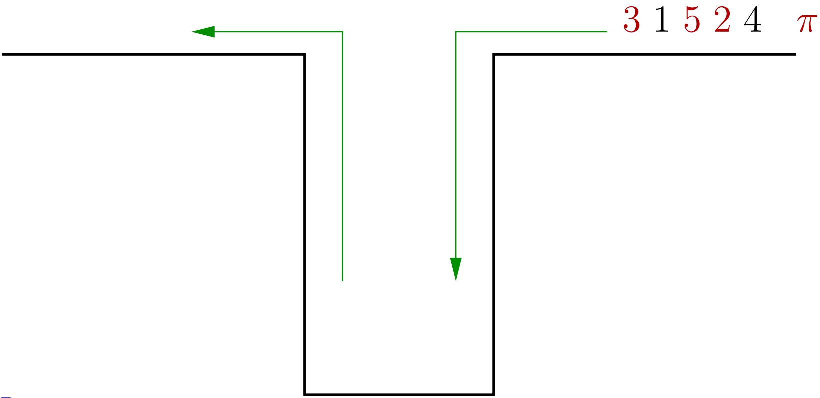

Proposition

A permutation π can be sorted by a stack if and only if it does not contain a subsequence $\dots a \dots b \dots c \dots$ with $c < a < b$.

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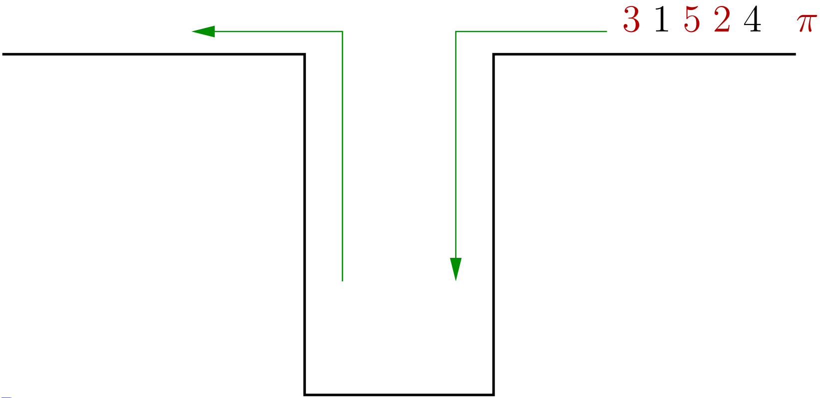

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i.e., such *"patterns"* *abc* (like 352) must be avoided

Permutations

A permutation $\pi \in S_n$ (bijection $\pi \in [n]^{[n]}$, $[n] := \{1, \dots, n\}$) will be considered as a word (n-tuple $\pi \in [n]^n$) of length n:

$$\pi_1 \dots \pi_n := (\pi(1), \dots, \pi(n)).$$

e.g. $\pi = 31524 \in [5]^5$ is the permutation
$1 \mapsto 3,\ 2 \mapsto 1,\ 3 \mapsto 5,\ 4 \mapsto 2,\ 5 \mapsto 4$

graphical representation:

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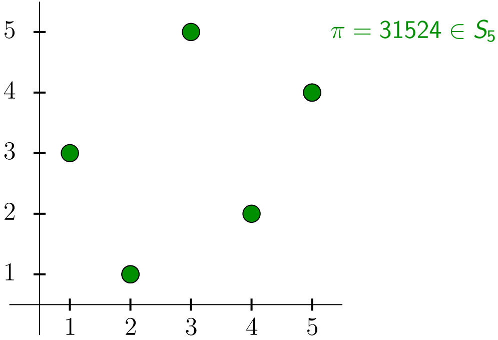
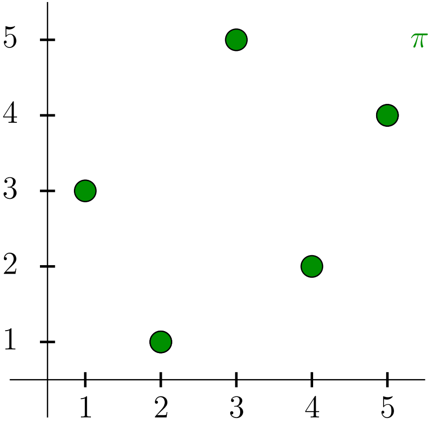


Permutation patterns (Example)



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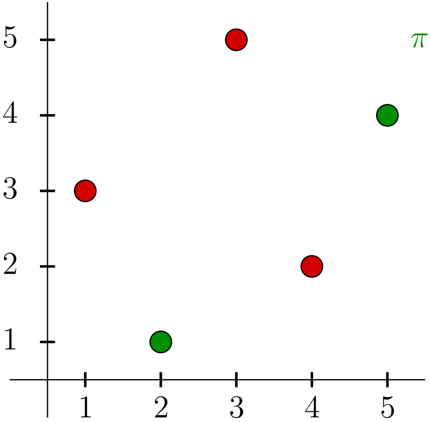

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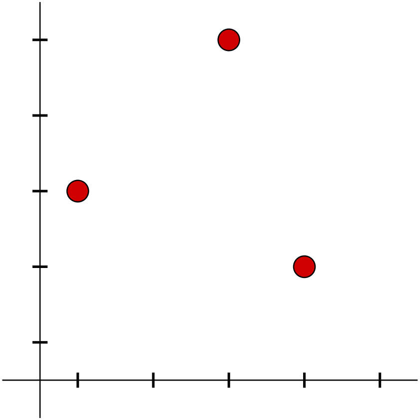

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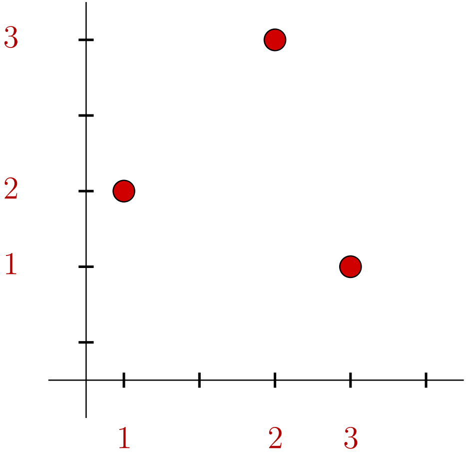

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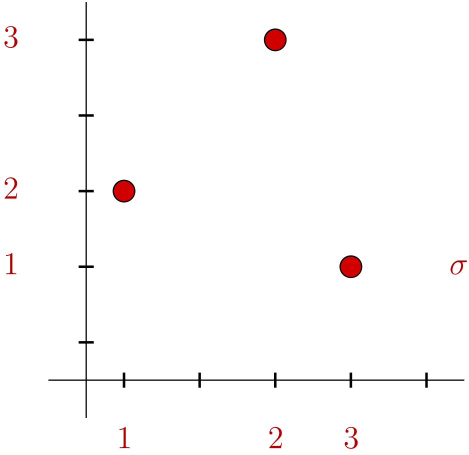

$\sigma = 231 \in S_3$

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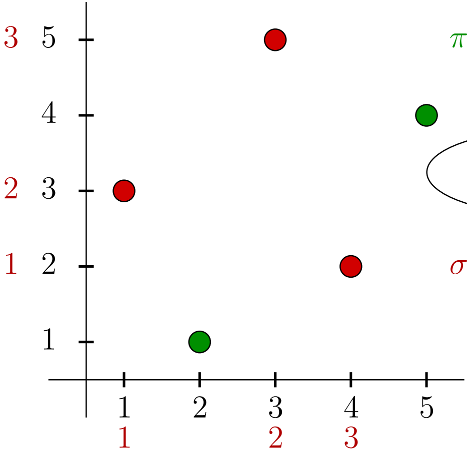


$\pi = 31524 \in S_5$

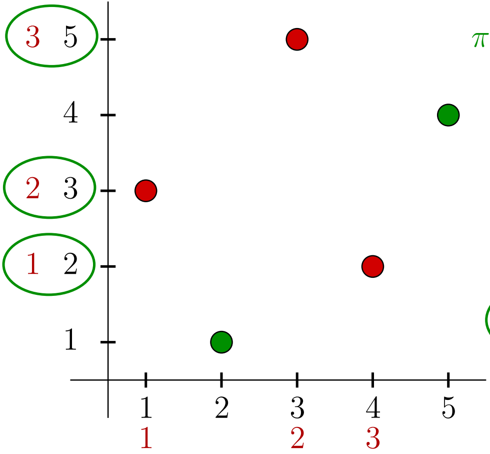
π involves σ

$\sigma \leq \pi$

σ is pattern of π

$\sigma = 231 \in S_3$

Permutation patterns (Example)



$\pi = 31524 \in S_5$

352 substring

231 reduced form

$231 = \mathrm{red}(352)$

Permutation patterns

$\ell \leq n$, $\sigma \in S_\ell$, $\pi \in S_n$

$\sigma \leq \pi :\Longleftrightarrow$ there exists a substring $\mathbf{u}$ of π of length ℓ such that $\sigma = \mathrm{red}(\mathbf{u})$

(σ is *ℓ-pattern* of π, or π *involves* σ)

π *avoids* $\sigma :\Longleftrightarrow \sigma \not\leq \pi$.

$\mathrm{Pat}^{(\ell)}\,\pi := \{\sigma \in S_\ell \mid \sigma \leq \pi\}$

The pattern involvement relation $\leq$ is a partial order on the set $\mathbb{P} := \bigcup_{n\geq 1} S_n$ of all finite permutations.

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The (monotone) Galois connection $\mathrm{Pat}^{(\ell)} - \mathrm{Comp}^{(n)}$

The relation $\sigma \not\leq \pi$ (π *avoids* σ) induces a Galois connection between subsets of S_ℓ and S_n. The corresponding "monotone Galois connection" (residuation) is given by the following operators (monotone w.r.t. $\subseteq$):

For $S \subseteq S_\ell$, $T \subseteq S_n$ ($\ell \leq n$) let

$$\mathrm{Comp}^{(n)} S := \{\tau \in S_n \mid \mathrm{Pat}^{(\ell)} \tau \subseteq S\} = \{\tau \in S_n \mid \forall \sigma' \in S_\ell \setminus S : \sigma' \not\leq \tau\},$$

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In particular we have the defining property of a monotone Galois connection:

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Proposition

If S is a subgroup of S_ℓ, then $\mathrm{Comp}^{(n)} S$ is a subgroup of S_n.

(The converse does not hold)

Sketch of the proof.

Assume that $S \leq S_\ell$. Let $\pi, \tau \in \mathrm{Comp}^{(n)} S$.
Thus $\mathrm{Pat}^{(\ell)} \pi, \mathrm{Pat}^{(\ell)} \tau \subseteq S$.
Crucial observation:

$$\mathrm{Pat}^{(\ell)} \pi^{-1} = (\mathrm{Pat}^{(\ell)} \pi)^{-1} := \{\sigma^{-1} \mid \sigma \in \mathrm{Pat}^{(\ell)} \pi\},$$

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Consequently, π^{-1} and $\pi\tau$ also belong to $\mathrm{Comp}^{(n)} S$ (since $S \leq S_\ell$). Thus $\mathrm{Comp}^{(n)} S$ is a group. □

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Modification of Comp – Pat
for permutation *groups* $G \leq S_\ell$ and $H \leq S_n$:

$$\mathrm{gComp}^{(n)}\, G := \langle \mathrm{Comp}^{(n)}\, G \rangle = \mathrm{Comp}^{(n)}\, G = \{\tau \in S_n \mid \mathrm{Pat}^{(\ell)}\,\tau \subseteq G\},$$

$$\mathrm{gPat}^{(\ell)}\, H := \langle \mathrm{Pat}^{(\ell)}\, H \rangle = \langle \bigcup_{\tau \in H} \mathrm{Pat}^{(\ell)}\,\tau \rangle,$$

This gives also a monotone Galois connection since we have:

$$\boxed{\mathrm{gPat}^{(\ell)}\, H \subseteq G \iff H \subseteq \mathrm{gComp}^{(n)}\, G}$$

Thus

$\mathrm{gPat}^{(\ell)}\,\mathrm{gComp}^{(n)}\, G \subseteq G$ (kernel operator),

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$$\begin{aligned}
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&H \subseteq \mathrm{gComp}^{(n)}\, \mathrm{gPat}^{(\ell)}\, H && \text{(closure operator)},\\
&\mathrm{gComp}^{(n)}\, G = \mathrm{gComp}^{(n)}\, \mathrm{gPat}^{(\ell)}\, \mathrm{gComp}^{(n)}\, G && \text{(closures)},\\
&\mathrm{gPat}^{(\ell)}\, H = \mathrm{gPat}^{(\ell)}\, \mathrm{gComp}^{(n)}\, \mathrm{gPat}^{(\ell)}\, H && \text{(kernels)}.
\end{aligned}$$

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The (monotone) Galois connection $\mathrm{gPat}^{(\ell)} - \mathrm{gComp}^{(n)}$

Modification of Comp – Pat
for permutation *groups* $G \leq S_\ell$ and $H \leq S_n$:

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The "Galois" connections $\mathrm{Pat}^{(\ell)} - \mathrm{Comp}^{(n)}$ and $\mathrm{gPat}^{(\ell)} - \mathrm{gComp}^{(n)}$

The Galois closures $\mathrm{gComp}^{(n)}\, G$ and Galois kernels $\mathrm{gPat}^{(\ell)}\, H$

Remarks and references

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How to characterize the Galois closures and the Galois kernels of the monotone Galois connection $\mathrm{gComp}^{(n)} - \mathrm{gPat}^{(\ell)}$ $(\ell < n)$?

Answer: as automorphism groups of special relations (*pc-relations* and *pc-extended invariants, resp.*)

Recall the (usual) Galois connection Aut – Inv (between permutations $\pi \in S_n$ and relations $\varrho \in \mathrm{Rel}_n$ on $\{1, \dots, n\}$):

$$\mathrm{Aut}\, R := \{\pi \in S_n \mid \forall \varrho \in R : \pi \triangleright \varrho\} \text{ for } R \subseteq \mathrm{Rel}_n,$$
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pc-relations

For $I \in \mathcal{P}_\ell(n)$ (ℓ-element subsets of $[n]$), let $h_I\colon [\ell] \to I$ be the order-isomorphism $([\ell], \leq) \to (I, \leq)$.

Example $\ell = 3$, $n = 5$, $I := \{3,5,2\} = \{2,3,5\} \in \mathcal{P}_3(5)$:

$$h_I : 1 \mapsto 2,\ 2 \mapsto 3,\ 3 \mapsto 5.$$

Thus, for $\mathbf{s} := (3,1)$, $\mathbf{r} := (5,2)$, we get $h_I(\mathbf{s}) = \mathbf{r}$ and $h_I^{-1}(\mathbf{r}) = \mathbf{s}$.

For $k \leq \ell \leq n$, $\varrho \subseteq [n]^k$, $\sigma \subseteq [\ell]^k$ define $\varrho^\vee \subseteq [\ell]^k$ and $\sigma^\wedge \subseteq [n]^k$ as

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$\varrho \subseteq [n]^k$ is called *pattern closed relation* (*pc-relation*) if $\varrho^{\vee\wedge} = \varrho$.

For $k = \ell$ this means $\mathbf{r} \in \varrho \wedge \mathrm{red}(\mathbf{r}) = \mathrm{red}(\mathbf{s}) \implies \mathbf{s} \in \varrho$.

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Characterization of the Galois closures

The Galois closures of the closure operator $\mathrm{gComp}^{(n)}\,\mathrm{gPat}^{(\ell)}$ can be characterized by a single irreflexive k-ary pc-relation.

We have:

Theorem

(A) $\mathrm{gComp}^{(n)}\,\mathrm{gPat}^{(\ell)}\,H = \mathrm{Aut}\,\mathrm{pcInv}\,H$ *for* $H \leq S_n$.

(B) *Let H be a subgroup of S_n. Then the following are equivalent:*

- (a) *H is Galois closed, i.e.,* $H = \mathrm{gComp}^{(n)}\,\mathrm{gPat}^{(\ell)}\,H$,
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Let $G \leq S_\ell$. A relation $\sigma \subseteq [\ell]^k$ ($k \leq \ell$) is called a *pattern closed extended invariant (pc-extended invariant) of G* if

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Remarks and references

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$$G, \mathrm{Comp}^{(\ell+1)}\,G, \ldots, \mathrm{Comp}^{(n)}\,G, \mathrm{Comp}^{(n+1)}\,G, \ldots$$

in the group case, i.e., $G \leq S_n$ (then $\mathrm{Comp}^{(n)}\,G = \mathrm{gComp}^{(n)}\,G$), and asked for the "asymptotic" behaviour of the above sequence.

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Thank you for your ATTENTION!