Diagram induced topological properties of congruence lattices

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Problem. For a given class $\mathcal{K}$ of algebras describe Con $\mathcal{K}$ =all lattices isomorphic to Con A for some $A \in \mathcal{K}$.

Or, at least,

for given classes $\mathcal{K}$, $\mathcal{L}$ determine if Con $\mathcal{K} =$ Con $\mathcal{L}$
and, if Con $\mathcal{K} \nsubseteq$ Con $\mathcal{L}$, determine

$$\mathrm{Crit}(\mathcal{K}, \mathcal{L}) = \min\{\mathrm{card}(L_c) \mid L \in \mathrm{Con}\,\mathcal{K} \setminus \mathrm{Con}\,\mathcal{L}\}$$

($L_c =$ compact elements of L)

Some critical points

We are interested in the case when $\mathcal{K}$ and $\mathcal{L}$ are (congruence-distributive) varieties. For instance,
$\mathrm{Crit}(\mathbf{N}_5, \mathbf{M}_3) = 5$,
$\mathrm{Crit}(\mathbf{M}_3, \mathbf{N}_5) = \mathrm{Crit}(\mathbf{M}_3, \mathbf{D}) = \aleph_0$,
$\mathrm{Crit}(\mathbf{M}_4, \mathbf{M}_3) = \aleph_2$,
$\mathrm{Crit}(\mathbf{Maj}, \mathbf{Lat}) = \aleph_2$.
($\mathbf{N}_5$, $\mathbf{M}_3$, $\mathbf{M}_4$ are well-known lattice varieties, $\mathbf{Lat}$ = all lattices, $\mathbf{Maj}$ = all majority algebras.)

Upper bound on critical points

Theorem

(Gillibert) *If* $\mathcal{K}$, $\mathcal{L}$ *are finitely generated congruence-distributive varieties, then* $\mathrm{Crit}(\mathcal{K}, \mathcal{L}) \leq \aleph_2$ *or* $\mathrm{Con}\,\mathcal{K} = \mathrm{Con}\,\mathcal{L}$.

All examples with critical point $\aleph_2$ are essentially based on the fact that the intesection of two compact congruences need not be compact.

CIP varieties

We say that a variety $\mathcal{K}$ has *Compact Intersection Property (CIP)* if $\mathrm{Con}_c\, A$ is a lattice for every $A \in \mathcal{K}$.

Examples: distributive lattices, Stone algebras, vector spaces ...

Theorem

(Baker, Blok, Pigozzi) *A finitely generated congruence-distributive variety $\mathcal{K}$ has CIP iff every subalgebra of a subdirectly irreducible algebra in $\mathcal{K}$ is subdirectly irreducible (or one-element).*

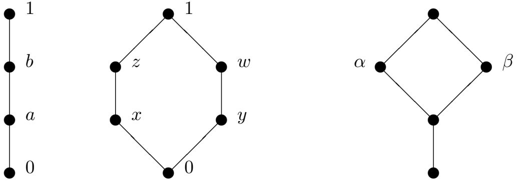
Upper bound for CIP

Theorem

If $\mathcal{K}$, $\mathcal{L}$ are finitely generated congruence-distributive varieties with CIP, then $\mathrm{Crit}(\mathcal{K}, \mathcal{L}) \leq \aleph_1$ *or* $\mathrm{Con}\,\mathcal{K} = \mathrm{Con}\,\mathcal{L}$.

The upper bound case can occur.

$\aleph_1$ example



C_4 N_6 Con $C_4 \cong$ Con N_6

Additional operations

Let C_4 be the 4-element chain depicted above endowed with an additional unary operation f defined by $f(0) = 0$, $f(a) = b$, $f(b) = a$, $f(1) = 1$. So, $C_4 = (\{a, b, 0, 1\}; \wedge, \vee, f, 0, 1)$. Let $\mathcal{C}_4$ be the variety generated by C_4.
Similarly, let N_6 be the bounded lattice depicted above, endowed with *two* additional unary operations g (180° rotation) and h (vertical symmetry). So, $N_6 = (\{x, y, z, w, 0, 1\}; \wedge, \vee, g, h, 0, 1)$. Let $\mathcal{N}_6$ be the variety generated by N_6.

Theorem

$\mathrm{Crit}(\mathcal{N}_6, \mathcal{C}_4) = \aleph_1$.

Method

Finding a link between two approaches in investigation of congruence lattices:

- liftability of semilattice diagrams;
- topological representation of distributive semilattices.

Con_c functor

For any homomorphism of algebras $f: A \to B$ we define

$$\mathrm{Con}_c f: \mathrm{Con}_c A \to \mathrm{Con}_c B$$

by
$\alpha \mapsto$ congruence generated by $\{(f(x), f(y)) \mid (x, y) \in \alpha\}$.

Fact. $\mathrm{Con}_c f$ preserves $\vee$ and 0, not necessarily $\wedge$.

For every commutative diagram $\mathcal{A}$ of algebras we have a commutative diagram $\mathrm{Con}_c \mathcal{A}$ of $(\vee, 0)$-semilattices.

Lifting of semilattice morphisms

Let

- $\varphi\colon\ S \to T$ be a homomorphism of $(\vee, 0)$-semilattices;
- $f\ :\ A \to B$ be a homomorphisms of algebras.

We say that f *lifts* φ, if there are isomorphisms $\psi_1\colon\ S \to \operatorname{Con}_c A$, $\psi_2\colon\ T \to \operatorname{Con}_c B$ such that

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & T \\ \psi_1 \downarrow & & \psi_2 \downarrow \\ \operatorname{Con}_c A & \xrightarrow{\operatorname{Con}_c f} & \operatorname{Con}_c B \end{array}$$

commutes.

More generally: lifting of commutative diagrams

Results of P. Gillibert

Let $\mathcal{K}$, $\mathcal{L}$ be locally finite congruence-distributive varieties.

Theorem

TFAE

- $\operatorname{Con}\mathcal{K} \nsubseteq \operatorname{Con}\mathcal{L}$;
- *there exists a diagram of finite* $(\vee, 0)$*-semilattices indexed by* $\{0,1\}^n$ *(for some* n*) liftable in* $\mathcal{K}$ *but not in* $\mathcal{L}$

Priestley duality

Let A belong to a finitely generated congruence-distributive variety with CIP. Then $\mathrm{Con}_c\, A$ is a distributive lattice and we can consider its dual Priestley space $\mathbf{X}$. (We use the version for lattices with 0 but not necessarily with 1.)

- Points of the dual space correspond to (completely) meet-irreducible elements of $\mathrm{Con}\, A$, so $\mathbf{X} = \mathrm{M}(\mathrm{Con}\, A)$.
- The order is inherited from $\mathrm{Con}\, A$.
- The topology is generated by the sets

$$M_{x,y} = \{\alpha \mid (x, y) \in \alpha\}$$

and their complements.

Diagrams versus topology

Theorem

Let $\mathcal{V}$ be a finitely generated congruence-distributive variety with CIP. Let F be the free algebra in $\mathcal{V}$ with $\aleph_1$ generators. Let $\vec{S}$ be a diagram of finite distributive $(0, \vee)$-semilattices indexed by a finite ordered set P having a smallest element $0 \in P$. The following conditions are equivalent.

(i) *There exists $A \in \mathcal{V}$ such that $\mathbf{X} = \mathrm{M}(\mathrm{Con}\, A)$ is $\vec{S}$-nonseparable;*

(ii) *$\mathrm{M}(\mathrm{Con}\, F)$ is $\vec{S}$-nonseparable;*

(iii) *$\vec{S}$ has a lifting in $\mathcal{V}$.*

What is $\vec{S}$-separability?

A simple example, instead of definition.
Let $\vec{S}$ be a diagram of $(\vee, 0)$-semilattices consisting of a single morphism $\varphi : \{0\} \to \{0, 1\}$.

- $\vec{S}$ in not liftable in the variety of Boolean algebras. Consequence: the largest element in the dual space (corresponding to 1-element congruence lattice) does not belong to the topological closure of the rest (points corresponding to 2-element lattices).
- $\vec{S}$ is liftable in the variety of distributive lattices. Consequence: the largest element in the dual space (corresponding to 1-element congruence lattice) *can* belong to the topological closure of the rest (points corresponding to 2-element lattices).

How $\aleph_1$ got there

Well known free set theorem:

Theorem

(Hajnal) *If* $|X| \geq \aleph_1$, *then for every function* $\Phi : X \to [X]^{<\omega}$ *there is a set* $Y \subseteq X$ *such that* $|Y| = |X|$ *and* $x \notin \Phi(y)$ *whenever* $x, y \in Y$, $x \neq y$.

Free set theorem for diagrams

Let $(A_p \mid p \in P)$ be a family of nonempty sets, indexed by a finite poset P with a smallest element, such that $A_p \subseteq A_q$ whenever $p \leq q$. For a set X let $H(X, A_p)$ denote the set of all *surjective* mappings $X \to A_p$

Theorem

If $|X| \geq \aleph_1$, then for every function

$$\mathrm{supp} : \bigcup_{p \in P} H(X, A_p) \to [X]^{<\omega}$$

there are $h_p \in H(X, A_p)$ such that

$$h_q \restriction \mathrm{supp}\, h_p = h_p \restriction \mathrm{supp}\, h_p$$

for every $p < q$.

(A diagram version of the free set theorem.)

Thanks

Thank you for attention.

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