Algebras with a central semilattice operation

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Semilattice ordered algebras

$\mho$ - the variety of all algebras (A, Ω) of type $\tau \colon \Omega \to \mathbb{N}$
$\mathcal{V}$ - a subvariety of $\mho$

Definition

An algebra $(A, \Omega, +)$ is called a *semilattice ordered $\mathcal{V}$-algebra* (SLO-algebra), if $(A, +)$ is a (join) semilattice, $(A, \Omega) \in \mathcal{V}$ and the operations from the set Ω distribute over the operation $+$.

The operation ω distributes over $+$ if for any $x_1, \dots, x_i, y_i, \dots, x_n \in A$

$$\omega(x_1, \dots, x_i + y_i, \dots, x_n) = \\ \omega(x_1, \dots, x_i, \dots, x_n) + \omega(x_1, \dots, y_i, \dots, x_n),$$

for any $1 \leq i \leq n$

Examples

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- Extended power algebras: $(\mathcal{P}_{>0}(A), \Omega, \cup)$
  (A, Ω) - an algebra
  $\mathcal{P}_{>0}(A)$ - the family of all non-empty subsets of A
  $\omega : \mathcal{P}_{>0}(A)^n \to \mathcal{P}_{>0}(A)$ - the complex operation:

$$\omega(A_1, \ldots, A_n) := \{\omega(a_1, \ldots, a_n) \mid a_i \in A_i\}$$

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- Modals = semilattice ordered modes (idempotent and entropic algebras)

$$\omega(x, \ldots, x) = x,$$
$$\omega(\phi(x_{11}, \ldots, x_{n1}), \ldots, \phi(x_{1m}, \ldots, x_{nm})) =$$
$$\phi(\omega(x_{11}, \ldots, x_{1m}), \ldots, \omega(x_{n1}, \ldots, x_{nm})),$$

for every m-ary $\omega \in \Omega$ and n-ary $\phi \in \Omega$

Semilattice modes = entropic modals

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Theorem [K.Kearnes]

To each variety $\mathcal{V}$ of semilattice modes one can associate a commutative semiring $\mathbf{R}(\mathcal{V})$, whose structure determines many of the properties of the variety.

The semiring associated with idempotent SLO algebras

$\mathcal{S}$ - a variety of idempotent SLO algebras $(S, \Omega, +)$
$F_{\mathcal{S}}(x, y)$ - the free $\mathcal{S}$-algebra on two generators x and y
$U(\mathcal{S}) = \{t \in F_{\mathcal{S}}(x, y) \mid t \geq y\}$

$$\bullet : F_{\mathcal{S}}(x, y) \times F_{\mathcal{S}}(x, y) \to F_{\mathcal{S}}(x, y); \quad (s, t) \mapsto s \bullet t(x, y) := s(t(x, y), y)$$

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The algebra $\mathbf{U}(\mathcal{S}) = (U(\mathcal{S}), \bullet, +, 1 = x + y, y)$ is a semiring satisfying $1 + r = 1$.

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Remark

Let $\mathcal{S}$ be a variety of modals. The semiring $\mathbf{U}(\mathcal{S})$ is commutative.

Free idempotent SLO algebras

$(F_{\mathcal{V}}(X), \Omega)$ - the free algebra over a set X in the variety $\mathcal{V} \subseteq \mathcal{U}$
$\mathcal{P}^{<\omega}_{>0} F_{\mathcal{V}}(X)$ - the set of all finite non-empty subsets of $F_{\mathcal{V}}(X)$
$\mathcal{S}_{\mathcal{V}}$ - the variety of semilattice ordered $\mathcal{V}$-algebras

Theorem (A.P. + A.Zamojska-Dzienio)

The semilattice ordered algebra $(\mathcal{P}^{<\omega}_{>0} F_{\mathcal{V}}(X), \Omega, \cup)$ is free over a set X in the variety $\mathcal{S}_{\mathcal{V}}$ if and only if $(\mathcal{P}^{<\omega}_{>0} F_{\mathcal{V}}(X), \Omega, \cup) \in \mathcal{S}_{\mathcal{V}}$.

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$\mathcal{V}$ - a linear variety (a variety defined by some linear identities)
The extended power algebra $(\mathcal{P}_{>0}^{<\omega} F_{\mathcal{V}}(X), \Omega, \cup)$ is free over a set X in the variety $\mathcal{S}_{\mathcal{V}}$.

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Theorem (A.P. + A.Zamojska-Dzienio)

$(\mathcal{P}_{>0}^{<\omega} F_{\mathcal{V}}(X), \Omega) \in \mathcal{V}$ *if and only if* $\mathcal{V}$ *is linear.*

Idempotent replica

$\mathcal{IV}$ - a linear and idempotent variety of Ω-algebras

Corollary

The idempotent replica of $(\mathcal{P}^{<\omega}_{>0}F_{\mathcal{IV}}(X), \Omega, \cup)$ *is free over a set* X *in the variety* $\mathcal{S}_{\mathcal{IV}}$.

The idempotent replica of $(\mathcal{P}^{<\omega}_{>0}F_{\mathcal{IV}}(X), \Omega, \cup)$ = the quotient algebra $(\mathcal{P}^{<\omega}_{>0}F_{\mathcal{IV}}(X)/\varrho, \Omega, \cup)$, where ϱ is the replica congruence relation

$$\varrho := \bigcap\{\phi \in Con(\mathcal{P}^{<\omega}_{>0}F_{\mathcal{IV}}(X), \Omega, \cup) \mid (\mathcal{P}^{<\omega}_{>0}F_{\mathcal{IV}}(X)/\phi, \Omega, \cup) \in \mathcal{S}_{\mathcal{IV}}\}$$

Example - Semilattice ordered semigroups

$(S, \cdot)$ - a semigroup
$S^1 := S \cup \{1\}$ (1 is a neutral element)

Definition

$A \subseteq S$ is 2-closed if for each $p, q \in S^1$ and $u_1, u_2 \in S$

$$pu_1q, pu_2q \in A \quad \Rightarrow \quad pu_1u_2q \in A.$$

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$\mathcal{ISG}$ - the variety of idempotent semigroups

Theorem (X.Z.Zhao)

Idempotent replica congruence of $(\mathcal{P}^{<\omega}_{>0} F_{\mathcal{ISG}}(X), \Omega, \cup)$*:*
for $Q, R \in \mathcal{P}^{<\omega}_{>0} F_{\mathcal{ISG}}(X)$

$$Q \varrho R \quad \Leftrightarrow \quad [Q]_2 = [R]_2.$$

$[X]_2$ = 2-closed subset generated by X

Example - Modals

$\mathcal{M}$ - a variety of Ω-modes

Theorem (A.Romanowska, J.D.H.Smith)

Idempotent replica congruence of $(\mathcal{P}_{>0}^{<\omega} F_{\mathcal{M}}(X), \Omega, \cup)$*: for* $Q, R \in \mathcal{P}_{>0}^{<\omega} F_{\mathcal{M}}(X)$

$$Q \alpha R \quad \Leftrightarrow \quad \langle Q \rangle = \langle R \rangle.$$

$\langle X \rangle$ = the subalgebra generated by X

$\mathcal{D}_{0,2}$ - the variety of differential groupoids

$$xx = x$$
$$xy \cdot uv = xu \cdot yv$$
$$xy \cdot z = xy.$$

$F_{\mathcal{D}_{0,2}}(x, y) = \{x, y, xy, yx\}$- the free differential groupoid on two generators x and y

$\cdot$	x	y	xy	yx
x	x	xy	x	xy
y	yx	y	yx	y
xy	xy	x	xy	x
yx	y	yx	y	yx

The idempotent replica of $(\mathcal{P}_{>0}^{<\omega} F_{\mathcal{D}_{0,2}}(x, y), \cdot, \cup)$ has 7 elements: $\{x\}/\alpha$, $\{y\}/\alpha$, $\{xy\}/\alpha$, $\{yx\}/\alpha$, $\{x, xy\}/\alpha$, $\{y, yx\}/\alpha$, $\{x, y\}/\alpha$.

The semiring $\mathbf{U}(\mathcal{S}_{\mathcal{D}_{0,2}})$

The semiring $\mathbf{U}(\mathcal{S}_{\mathcal{D}_{0,2}})$ associated with the variety $\mathcal{S}_{\mathcal{D}_{0,2}}$ has 3 elements:

- $1 = \{x, y, xy, yx\} = \{x\} + \{y\}$
- $\{y, yx\} = \{y\} + \{yx\}$
- $0 = \{y\}$

$\bullet$	0	$\{y, yx\}$	1
0	0	0	0
$\{y, yx\}$	0	0	$\{y, yx\}$
1	0	$\{y, yx\}$	1

SLO idempotent algebras with a central semilattice operation

Theorem [K.Kearnes]

The lattice of subvarieties of a variety of semilattice modes $\mathcal{V}$ is dually isomorphic to the congruence lattice $Con\mathbf{R}(\mathcal{V})$ of the semiring $\mathbf{R}(\mathcal{V})$ associated with $\mathcal{V}$.

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for every n-ary $\omega \in \Omega$

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- $\mathcal{S}_{\mathcal{J}\mathcal{V}}$ satisfies each identity which is true in $\mathcal{V}$
- $(A, \Omega, +)$ - a subdirectly irreducible $\mathcal{S}_{\mathcal{J}\mathcal{V}}$-algebra $\Rightarrow$ $(A, \Omega, +)$ has the least element 0.

Thank you for your attention.