Complexity classification for the binary branching semilinear-order constraint satisfaction problems

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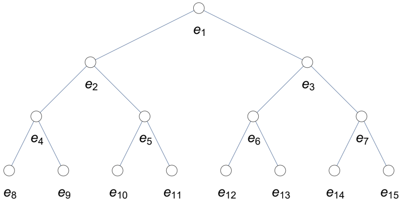
1. Constraint satisfaction problems on binary branching semi-linear order
2. BBS-SAT(Ψ) as a CSP
3. Complexity classification
   - Main result
   - Algebraic tools

semi-linear order

A partial order $(P; \leq)$ is called semilinear order if for any $a, b \in P$ the set $(\{x \in P : x \geq a \wedge x \geq b\}; \leq)$ is a linear order.



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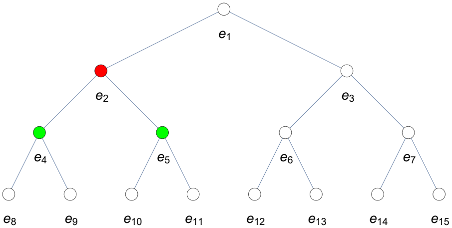
- below every element there are two incomparable elements.



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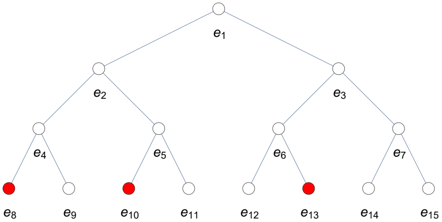
- below every element there are two incomparable elements.
- for any three incomparable elements



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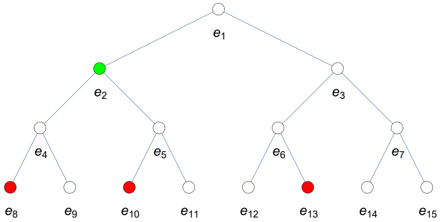
- below every element there are two incomparable elements.
- for any three incomparable elements there is an element of P such that it is greater than two of the three and incomparable to the third.



BBS-SAT(Ψ)

Let $\Psi := \{\psi_1, \psi_2, \ldots, \psi_k\}$ be a set of first-order formulas over language $\{\leq\}$. The constraint satisfaction problem BBS-SAT(Ψ) is defined as follows.

Instance: A set of variables V and a formula $\Phi := \phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_n$, where ϕ_i is in Ψ by substituting some variables from V.

Question: Is there some branching binary semilinear order that contains nodes V and satisfies the formula Φ?

Example

Let $\Psi := \{x > y \wedge x \perp z\}$, where $x \perp z := \neg(x \leq z \vee z \leq x)$. Consider an input $(u > w \wedge x \perp w) \wedge (u > y \wedge y \perp z) \wedge (v > z \wedge z \perp y)$.

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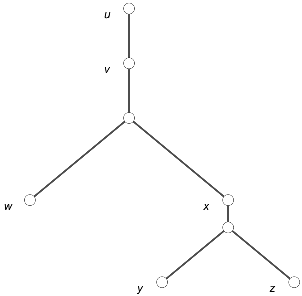

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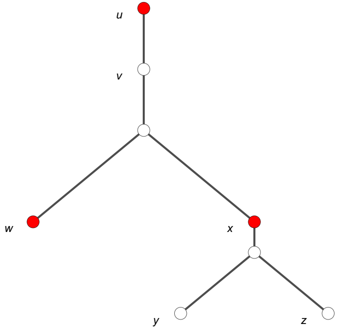

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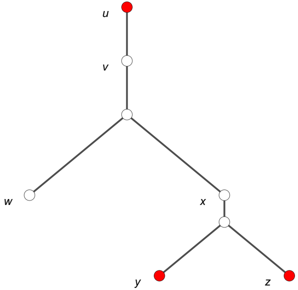

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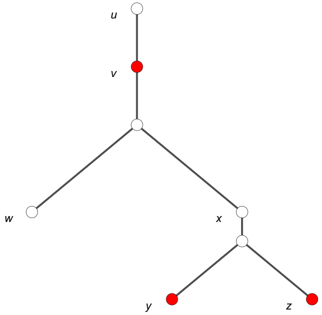

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- Some partial complexity results for branching time model were obtained by Broxval, Jonsson, Coppersmith and Winograd.
- We present here a complete complexity classification for branching time model.

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Proposition

There is a unique binary branching semilinear order $(\mathbb{S}_2; \leq)$ (up to isomorphism) that satisfies the following conditions.

- dense if for every $x, y \in \mathbb{S}_2$ such that $x < y$ there is $z \in \mathbb{S}_2$ such that $x < z < y$.
- unbounded if for every $x \in \mathbb{S}_2$ there are y, z such that $y < x < z$.
- without joins if for every $x, y \leq z$ and x, y incomparable, there is $u \in \mathbb{S}_2$ such that $x, y \leq u$ and $u < z$.

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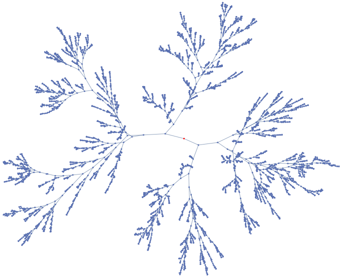

Important properties of $(\mathbb{S}_2; \leq)$

Every finite binary branching semilinear order is embedded into $(\mathbb{S}_2; \leq)$

From formulas to relations

Let $\Psi := \{\psi_1, \psi_2, \ldots, \psi_m\}$ be a set of first-order formulas over language $\{\leq\}$. For each ψ_i let $R_{\psi_i} := \{x \in \mathbb{S}_2^k : \psi_i(x)$ holds in $(\mathbb{S}_2; \leq)\}$, where k is the arity of ψ_i. The structure $(\mathbb{S}_2; R_{\psi_1}, R_{\psi_2}, \ldots, R_{\psi_m})$ is called a reduct of $(\mathbb{S}_2; \leq)$.

BBS-SATs as CSPs

The problem BBS-SAT(Ψ) can be reformulated as a CSP as follows.
Instance: A finite relational structure $\mathbb{A}$ over language $\{R_{\psi_1}, R_{\psi_2}, \ldots, R_{\psi_m}\}$.
Question: Is there a homomorphism from $\mathbb{A}$ to $(\mathbb{S}_2; R_{\psi_1}, R_{\psi_2}, \ldots, R_{\psi_m})$?

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Important relations

- $B := \{(x, y, z) \in \mathbb{S}_2^3 : x < y < z \vee z < y < x \vee x < y \wedge y \perp z \vee z < y \wedge x \perp y\}$.
- $N := \{(x, y, z) \in \mathbb{S}_2^3 : x|yz \vee z|xy\}$, where $x|yz := \exists t. x \perp t \wedge t > y \wedge t > z$.
- $T_3 := \{(x, y, z) \in \mathbb{S}_2^3 : x = y > z \vee x = z > y\}$.

Lemma

$\mathrm{CSP(B)}, \mathrm{CSP(N)}$ *and* $\mathrm{CSP}(\mathbb{T}_3)$ *are NP-complete.*

A Horn formula in $(\mathbb{S}_2; \leq)$ is a conjunction of the formulas of the form

$$x_1 \neq y_1 \vee x_2 \neq y_2 \vee \cdots \vee x_k \neq y_k \vee$$
$$\vee\, \mathcal{T}(z_1, z_2, \ldots, z_m) \vee \bigvee_{b \in B \setminus \{\underline{0}, \underline{1}\}} \{z_i : b_i = 0\} | \{z_i : b_i = 1\},$$

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$$x_1 \neq y_1 \vee x_2 \neq y_2 \vee x_k \neq y_k \vee$$
$$\mathcal{T}(z_1, z_2, \ldots, z_m) \wedge (z_1 > z_m \vee z_2 > z_m \vee \cdots \vee z_{m-1} > z_m)$$
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or of the form

$$x_1 \neq y_1 \vee x_2 \neq y_2 \vee x_k \neq y_k \vee z_1 = z_2 = \cdots = z_m$$
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Theorem (M. Bodirsky and T. V. Pham, 2016)

Let Γ be a reduct of $(\mathbb{S}_2; \leq)$*. Then one of the following applies.*

- $\mathrm{End}(\Gamma)$ *contains a function whose range induces a chain in* $(\mathbb{S}_2; \leq)$*, and* $\mathrm{CSP}(\Gamma)$ *is reduced to a CSP for a reduct of* $(\mathbb{Q}; \leq)$*.*

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- $\mathrm{End}(\Gamma) = \overline{\mathrm{Aut}(\mathbb{S}_2; \leq)}$ *and every relation in* Γ *can be defined by a Horn formula, and* $\mathrm{CSP}(\Gamma)$ *can be solved in polynomial time.*

Tools for complexity clasification

The following tools are used to classify the complexity of BBS-SAT(Ψ):

- Galois connection between Polymorphism clone and primitive positive definability of an ω-categorical structure (Bodirsky and Nešetřil).
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