# On products of amalgams and amalgams of products

Maja Pech

Institute of Algebra
TU Dresden
Germany

Department of Mathematics and Informatics
University of Novi Sad
Serbia

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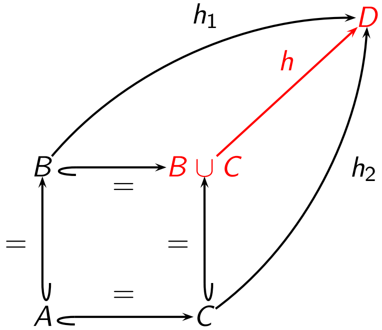
*joint work with Christian Pech*

# Canonical amalgams

Example: Sets

- Given $B$ and $C$ with $B \cap C = A$.

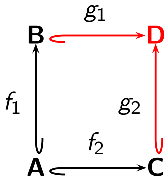



- For all $D$, and $h_1, h_2$ with $h_1\restriction_A = h_2\restriction_A$, there is a unique $h : B \cup C \to D$, such that $h\restriction_B = h_1$ and $h\restriction_C = h_2$:

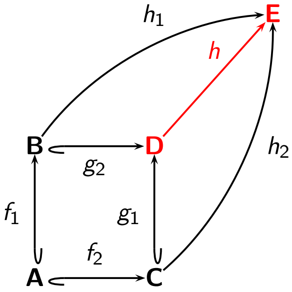
$$h(x) = \begin{cases} h_1(x), & x \in B, \\ h_2(x), & x \in C. \end{cases}$$

# Categorification

Let $\mathcal{C}$ be a category. A commuting square



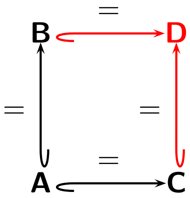

is called a **pushout square** if for all $\mathbf{E} \in \mathcal{C}$ and all $h_1 : \mathbf{B} \to \mathbf{E}$, $h_2 : \mathbf{C} \to \mathbf{E}$ with $h_1 \circ f_1 = h_2 \circ f_2$ there is a unique $h : \mathbf{D} \to \mathbf{E}$ s.t. the following diagram commutes:



# Amalgamated free-sums

- Given a class $\mathcal{K}$ of relational structures of the same type.
- $\mathcal{K}$ together with homomorphisms can be considered as a category.

Given $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ such that $\mathbf{A} \leq \mathbf{B}$, $\mathbf{A} \leq \mathbf{C}$.
A structure $\mathbf{D} \in \mathcal{K}$ is called **amalgamated free-sum** of $\mathbf{B}$ and $\mathbf{C}$ (w.r.t. to $\mathbf{A}$) if $\mathbf{B}, \mathbf{C} \leq \mathbf{D}$ and




is a pushout square in $\mathcal{K}$.
Notation: We denote $\mathbf{D}$ by $\mathbf{B} \oplus_{\mathbf{A}} \mathbf{C}$.

# Strict amalgamation classes

A class $\mathcal{K}$ of finite relational structures is called a **strict amalgamation class** if

- it has (**HP**) and (**JEP**), and
- it is closed with respect to amalgamated free-sums.

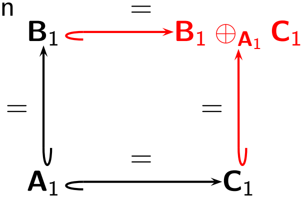
Examples: finite simple graphs, finite posets.

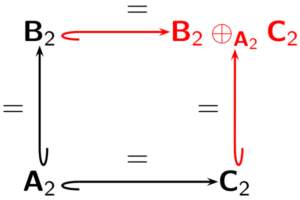
Non-example: finite chains.

# Amalgams of products and products of amalgams

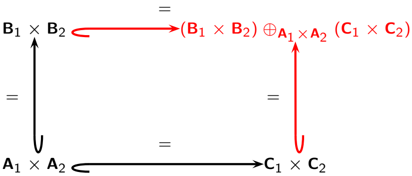
Let $\mathcal{K}$ be a strict amalgamation class that is closed w.r.t. products.

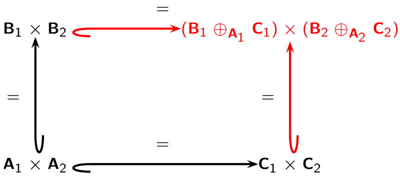
Given



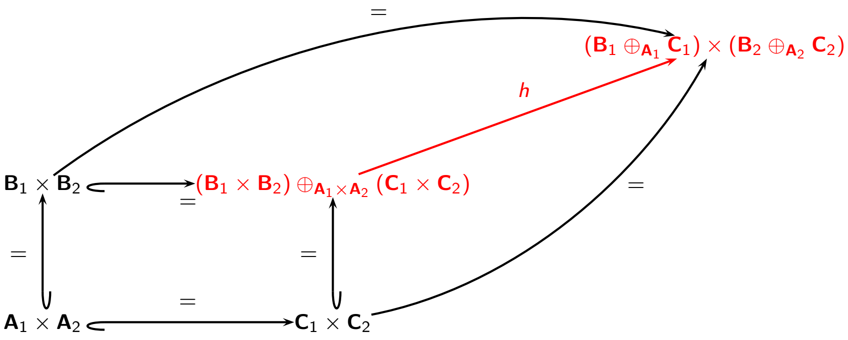

There are two different ways to amalgamate the products:




How are they related?

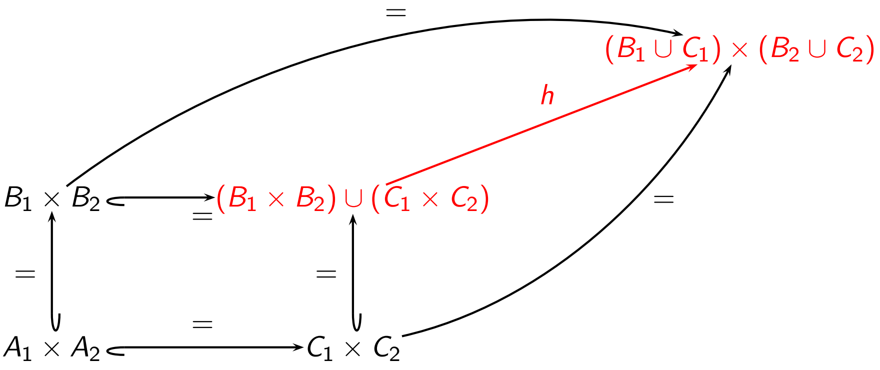
# Connection between products and amalgamated free sums




What is $h$? Is it an embedding?

# Example 1: Sets

amalgamated free-sum of sets = set union.

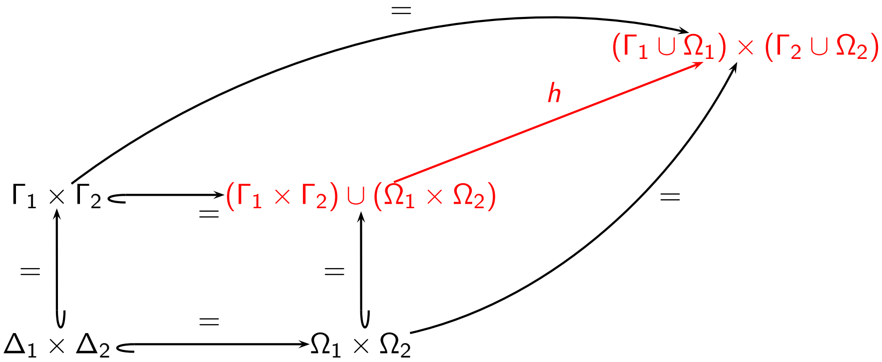



$$(B_1 \cup C_1) \times (B_2 \cup C_2) = (B_1 \times B_2) \cup (B_1 \times C_2) \cup (C_1 \times B_2) \cup (C_1 \times C_2)$$

$\Rightarrow h$ is an embedding.

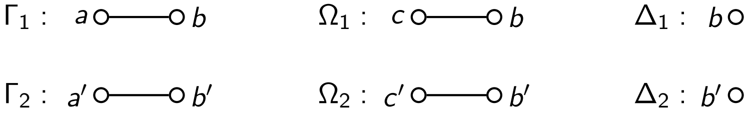
# Non-example 1: Graphs

amalgamated free-sum on graphs = union of graphs.



# Non-example 1: Graphs (cont.)

Consider the following graphs:




Then

- $((a, b'), (b, c')) \in E((\Gamma_1 \cup \Omega_1) \times (\Gamma_2 \cup \Omega_2))$,
- $((a, b'), (b, c')) \notin E((\Gamma_1 \times \Gamma_2) \cup (\Omega_1 \times \Omega_2))$.

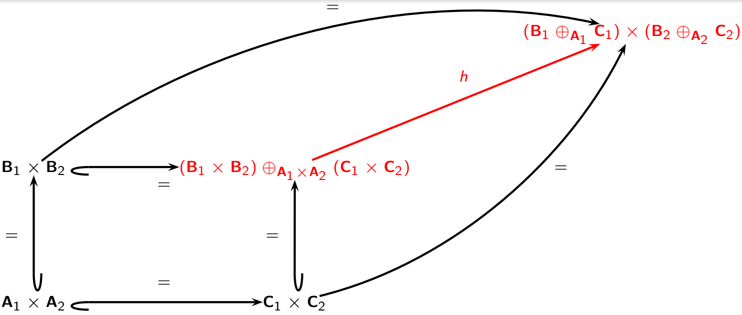
*h* is not an embedding.

# Example 2: Posets

## Amalgamated free-sum of posets

Given $\mathbf{B} = (B, \leq_{\mathbf{B}})$ and $\mathbf{C} = (C, \leq_{\mathbf{C}})$.

- $\mathbf{A} = (A, \leq_{\mathbf{A}})$, where $A = B \cap C$, $\leq_{\mathbf{A}} = \leq_{\mathbf{B}}\restriction_A = \leq_{\mathbf{C}}\restriction_A$.
- $\mathbf{B} \oplus_{\mathbf{A}} \mathbf{C} = (B \cup C, \leq_{\oplus})$, where $\leq \oplus$ is the transitive closure of $\leq_{\mathbf{B}} \cup \leq_{\mathbf{C}}$.




$h$ is always an embedding.

# Non-example 2: Metric spaces with non-expansive maps

## Amalgamated free-sum of metric spaces

Given non-empty finite metric spaces $(B, d_B)$, $(C, d_C)$ and $(A, d_A)$, where $A = B \cap C$, $d_A = d_B\restriction_A = d_C\restriction_A$.
$(B, d_B) \oplus_{(A,d_A)} (C, d_C) = (B \cup C, d_\oplus)$, where

$$d_\oplus(x,y) = \begin{cases} d_B(x,y), & x,y \in B \\ d_C(x,y), & x,y \in C \\ \min_{z\in A}(d_B(x,z) + d_C(z,y)), & x \in B, y \in C \\ \min_{z\in A}(d_C(x,z) + d_B(z,y)), & x \in C, y \in B. \end{cases}$$

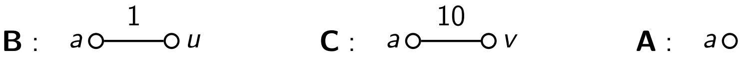
## Product of metrics

$(B, d_B) \times (C, d_C) = (B \times C, d_\times)$, where

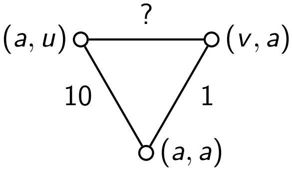
$$d_\times((b_1,c_1),(b_2,c_2)) = \max\{d_B(b_1,b_2), d_C(c_1,c_2)\}.$$

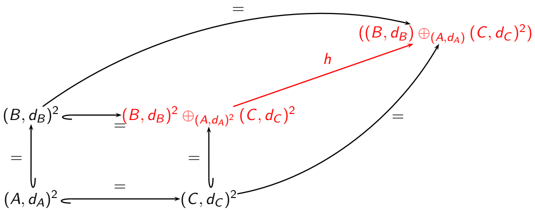
# Non-example 2: Metric spaces with non-expansive maps (cont.)

Given




Consider




- $d_{\oplus}((a,u),(v,a)) = 11$
- $d_{\times}((a,u),(v,a)) = 10$

*h* is an injective contraction, not an embedding.

# Example 3: Ultrametric spaces with non-expansive maps

## Amalgamated free-sum of ultrametric spaces

Given non-empty finite ultrametric spaces $(B, d_B)$, $(C, d_C)$ and $(A, d_A)$, where $A = B \cap C$, $d_A = d_B\restriction_A = d_C\restriction_A$.
$(B, d_B) \oplus_{(A, d_A)} (C, d_C) = (B \cup C, d_\oplus)$, where

$$d_\oplus(x, y) = \begin{cases} d_B(x, y), & x, y \in B \\ d_C(x, y), & x, y \in C \\ \min_{z\in A} \max\{d_B(x, z), d_C(z, y)\}, & x \in B, y \in C \\ \min_{z\in A} \max\{d_C(x, z), d_B(z, y)\}, & x \in C, y \in B. \end{cases}$$

## Product of ultrametrics

$(B, d_B) \times (C, d_C) = (B \times C, d_\times)$, where

$$d_\times((b_1, c_1), (b_2, c_2)) = \max\{d_B(b_1, b_2), d_C(c_1, c_2)\}.$$

*h* is always an embedding.

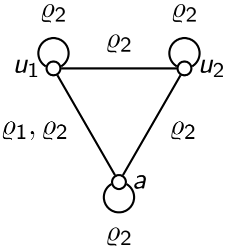
# Non-example 3: "Almost" ultrametrics

Given binary relations $\varrho_1$, $\varrho_2$ with

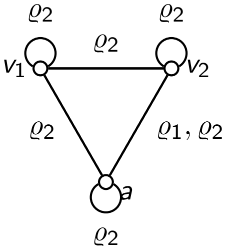
(1) $\varrho_2$ is reflexive,
(2) $\varrho_1$, $\varrho_2$ are symmetric
(3) $\varrho_1(x,y) \wedge \varrho_2(y,z) \implies \varrho_2(x,z)$

Consider

**B** :




**C** :




- amalgamated free-sum = union closed w.r.t. (1) – (3)
- $\mathbf{B}^2 \oplus_{\mathbf{A}^2} \mathbf{C}^2$: $(u_1, u_2)$ and $(v_1, v_2)$ are not related.
- $(\mathbf{B} \oplus_{\mathbf{A}} \mathbf{C})^2$: $\varrho_2((u_1,u_2),(v_1,v_2))$.

*h* is not an embedding.

# Homework

- Give a useful sufficient condition on strict amalgamation classes, so that the canonical homomorphisms from amalgams of products to products of amalgams are always embeddings.
- Extra credits are given for a useful necessary and sufficient condition.

Written solutions will be collected, corrected and evaluated at AAA93.