Characterizations of Idempotent Elements in Pre-Generalized Hypersubstitutions of type $\tau = (m, n)$

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Outline

Motivation

In 2000, W. Puninagool and S. Leeratanavalee [10]

1. studied *pre-generalized hypersubstitutions* of type τ
2. determined idempotent elements of type $(2, 2)$.

We characterize the idempotent of type (m, n).

[10] Puninagool W. and Leeratanavalee S., *Idempotent pre-generalized hypersubstitutions of type* $\tau = (2, 2)$, Analele Stiintifice ale Universitatii Ovidius Constanta **15(2)** (2007), 55–70.

Generalized Superposition of Terms

The concept of *generalized superposition of terms* [8]

$$S^k : (W_\tau(X))^{k+1} \to W_\tau(X)$$

(i) if $t = x_j \in X_k$, then $S^k(x_j, t_1, \ldots, t_k) := t_j$;

(ii) if $t = x \in X \setminus X_k$, then $S^k(x, t_1, \ldots, t_k) := x$;

(iii) if $t = f_i(s_1, \ldots, s_{n_i})$ and assume that $S^k(s_j, t_1, \ldots, t_k)$ for $1 \leq j \leq n_i$ are already defined, then

$$S^k(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_k) := f_i(S^k(s_1, t_1, \ldots, t_k), \ldots, S^k(s_{n_i}, t_1, \ldots, t_k)).$$

[8] Leeratanavalee S. and Denecke K., *Generalized hypersubstitutions and strongly solid varieties*, General Algebra and Applications, Proc. of the “59$^{\text{th}}$ workshop on general algebra”, “15$^{\text{th}}$ Conference for young algebraists Potsdam 2000”, pp. 135–145, Shaker Verlag, (2000).

Generalized Hypersubstitutions

A *generalized hypersubstitution* of type τ is a mapping

$$\sigma : \{f_i : i \in I\} \to W_\tau(X)$$

-may not preserved arity

$\mathrm{Hyp}_{\mathrm{G}}(\tau)$–the set of all generalized hypersubstitutions of type τ

Generalized Hypersubstitutions

$$\hat{\sigma} : W_\tau(X) \to W_\tau(X)$$

(i) $\hat{\sigma}[x] := x \in X$;

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ for any n_i-ary operation symbol f_i and assume that $\hat{\sigma}[t_j]$ are already defined for all $1 \leq j \leq n_i$.

a binary operation $\circ_{\mathrm{G}}$ on $\mathrm{Hyp}_{\mathrm{G}}(\tau)$ is defined by

$$\sigma_1 \circ_{\mathrm{G}} \sigma_2 := \hat{\sigma}_1 \circ \sigma_2.$$

$\mathbf{Hyp}_{\mathrm{G}}(\tau) := (\mathrm{Hyp}_{\mathrm{G}}(\tau); \circ_{\mathrm{G}}, \sigma_{\mathrm{id}})$–monoid

[8] Leeratanavalee S. and Denecke K., *Generalized hypersubstitutions and strongly solid varieties*, General Algebra and Applications, Proc. of the "59$^{\mathrm{th}}$ workshop on general algebra", "15$^{\mathrm{th}}$ Conference for young algebraists Potsdam 2000", pp. 135–145, Shaker Verlag, (2000).

For $t, t_1, \ldots, t_n \in W_\tau(X)$ and $\sigma, \sigma_1, \sigma_2 \in \mathrm{Hyp_G}(\tau)$ we have

(i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, \ldots, t_n)]$,

(ii) $(\hat{\sigma}_1 \circ \sigma_2)\widehat{} = \hat{\sigma}_1 \circ \hat{\sigma}_2$.

$\mathbf{Hyp}(\tau) \leq \mathbf{Hyp}_{\mathrm{G}}(\tau)$

Pre-Generalized Hypersubstitutions of type (m, n)

- f, g–operation symbols of type (m, n)
- denote the generalized hypersubstitution σ with $\sigma(f) = t_1$ and $\sigma(g) = t_2$ by σ_{t_1,t_2}

Definition

A generalized hypersubstitution σ of type (m, n) is called a *pre-generalized hypersubstitution* if the terms $\sigma(f)$ and $\sigma(g)$ are not variables.

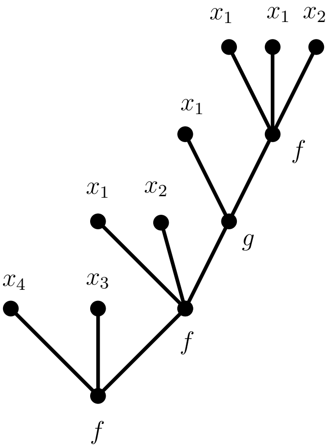
Preliminary

Let F be a variable over the two-elements alphabet $\{f, g\}$. Let $t = F(t_1, \ldots, t_j)$ where F has arity $j \in \{m, n\}$ and $i \leq \min\{m, n\}$, we define $M^i(t)$ by:

(i) if $t_i \in X$, then $M^i(t) = t_i$;

(ii) if $t_i = F'(s_1 \ldots, s_k)$ where F' has arity $k \in \{m, n\}$ and assume that $M^i(s_i)$ are already defined, then $M^i(t) = M^i(s_i)$.

Example



$M^1(t) = x_4$
$M^2(t) = x_3$
$M^3(t)$ does not define.

Main Results

Proposition

Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n). Then the following statements are equivalent:

(i) σ_{t_1,t_2} is idempotent;

(ii) $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$ and $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$.

Notations

- var(t)–the set of all variables occurring in the term t
- op(t)–the number of operation symbols occurring in the term t
- ops(t)–the set of all operation symbols occurring in the term t
- firstop(t)–the first operation symbol (from the left) occurring in the term t

Main Results

1. $\mathrm{op}(t_1) = 1$ and $\mathrm{op}(t_2) = 1$;
2. $\mathrm{op}(t_1) = 1$, $\mathrm{op}(t_2) > 1$ and
   $\mathrm{op}(t_1) > 1$, $\mathrm{op}(t_2) = 1$;
3. $\mathrm{op}(t_1) > 1$ and $\mathrm{op}(t_2) > 1$.

Case 1: $\mathrm{op}(t_1) = 1$ and $\mathrm{op}(t_2) = 1$

1. $\mathrm{firstop}(t_1) = f$ and $\mathrm{firstop}(t_2) = f$;
2. $\mathrm{firstop}(t_1) = g$ and $\mathrm{firstop}(t_2) = g$;
3. $\mathrm{firstop}(t_1) = f$ and $\mathrm{firstop}(t_2) = g$

since for the case $\mathrm{firstop}(t_1) = g$ and $\mathrm{firstop}(t_2) = f$ is impossible. Indeed, for $s_1, \ldots, s_m \in X$,

$$t_1 = \hat{\sigma}_{t_1,t_2}[t_1] = S^m(\sigma_{t_1,t_2}(g), s_1, \ldots, s_m) = S^m(t_2, s_1, \ldots, s_m).$$

This implies $\mathrm{firstop}(t_1) = f$, a contradiction.

Case 1: $\mathrm{op}(t_1) = 1$ and $\mathrm{op}(t_2) = 1$

Theorem

Let $t_1 = f(s_1, \ldots, s_m)$ and $t_2 = f(s'_1, \ldots, s'_m)$ where $s_1, \ldots, s_m, s'_1, \ldots, s'_m \in X$. Then the following statements are equivalent:

(1) σ_{t_1,t_2} is idempotent;

(2) the following conditions holds:

(i) if $x_j \in \mathrm{var}(t_1)$ where $1 \leq j \leq m$, then $s_j = x_j$;

(ii) if $s_i = x_j$ where $1 \leq i, j \leq m$, then $s'_i = s'_j$;

(iii) if $s_j = x \in X \setminus X_m$ where $1 \leq j \leq m$, then $s'_j = x$.

σ_{t_1,t_2}

1. $t_1 = f(x_1, x_1, x_4)$
   $t_2 = f(x_2, x_2, x_4)$
2. $t_1 = f(x_1, x_1, x_3)$
   $t_2 = f(x_1, x_2, x_3)$

Case 1: $\mathrm{op}(t_1) = 1$ and $\mathrm{op}(t_2) = 1$

Theorem

Let $t_1 = f(s_1, \ldots, s_m)$ and $t_2 = g(s'_1, \ldots, s'_n)$ where $s_1, \ldots, s_m, s'_1, \ldots, s'_n \in X$. Then the following statements are equivalents:

(1) σ_{t_1,t_2} is idempotent;

(2) the following conditions holds:

- (i) if $x_j \in \mathrm{var}(t_1)$ where $1 \leq j \leq m$, then $s_j = x_j$;
- (ii) if $x_k \in \mathrm{var}(t_2)$ where $1 \leq k \leq n$, then $s'_k = x_k$.

σ_{t_1,t_2}

1. $t_1 = f(x_4, x_2, x_3)$
   $t_2 = g(x_2, x_2)$
2. $t_1 = f(x_2, x_3, x_3)$
   $t_2 = g(x_2, x_1)$

Case 2: $\mathrm{op}(t_1) = 1$, $\mathrm{op}(t_2) > 1$ and $\mathrm{op}(t_1) > 1$, $\mathrm{op}(t_2) = 1$

Theorem

Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m, n), $\mathrm{op}(t_1) = 1$, $\mathrm{op}(t_2) > 1$, $t_1 = f(s'_1, \ldots, s'_m)$ and $t_2 = g(s_1, \ldots, s_n)$ where $s'_1, \ldots, s'_m \in X$, $s_1, \ldots, s_n \in W_{(m,n)}(X)$. Then the following conditions are equivalent:

(1) σ_{t_1,t_2} is idempotent;

(2) the following conditions hold

  (i) if $x_j \in \mathrm{var}(t_1)$ where $1 \leq j \leq m$, then $s'_j = x_j$;

  (ii) if $x_j \in \mathrm{var}(t_2)$ where $1 \leq j \leq n$, then $s_j = x_j$.

σ_{t_1,t_2}

1. $t_1 = f(x_4, x_2, x_2)$
   $t_2 = g(x_1, f(x_1, x_3, x_1))$
2. $t_1 = f(x_4, x_2, x_2)$
   $t_2 = g(x_2, f(x_1, x_3, x_1))$

Case 2: $\mathrm{op}(t_1) = 1$, $\mathrm{op}(t_2) > 1$ and $\mathrm{op}(t_1) > 1$, $\mathrm{op}(t_2) = 1$

Theorem

Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n), $\mathrm{op}(t_1) = 1$, $\mathrm{op}(t_2) > 1$, $t_1 = f(s_1,\ldots,s_m)$ and $t_2 = f(\overline{s}_1,\ldots,\overline{s}_m)$ where $s_1,\ldots,s_n \in X$, $\overline{s}_1,\ldots,\overline{s}_n \in W_{(m,n)}(X)$. Then the following conditions are equivalent:

(1) σ_{t_1,t_2} is idempotent;

(2) if $x_j \in \mathrm{var}(t_1)$ where $1 \leq j \leq m$, then $s_j = x_j$ and the following conditions hold:

  (i) if $s_j = x \in X \setminus X_m$ where $1 \leq j \leq m$, then $\overline{s}_j = x$ and for each $\overline{s}_i \notin X$ where $1 \leq i \leq m$, $M^j(\overline{s}_i) = x$;

  (ii) if $s_j = s_l$ where $1 \leq j,l \leq m$, then $\overline{s}_j = \overline{s}_l$;

  (iii) For each subterm of t_2 which is not a variable $f(s'_1,\ldots,s'_m)$ where $s'_1,\ldots,s'_m \in W_{(m,n)}(X)$, if $s_j = s_l$ where $1 \leq j,l \leq m$, then $s'_j = s'_l$.

σ_{t_1,t_2}

1. $t_1 = f(x_1, x_4, x_1)$
   $t_2 = f(f(x_2, x_4, x_2), x_4, f(x_2, x_4, x_2))$
2. $t_1 = f(x_1, x_4, x_1)$
   $t_2 = f(f(x_2, x_2, x_1), x_4, x_3)$

Case 3: $\mathrm{op}(t_1) > 1$ and $\mathrm{op}(t_2) > 1$

1. $\mathrm{firstop}(t_1) = f$ and $\mathrm{firstop}(t_2) = f$;
2. $\mathrm{firstop}(t_1) = g$ and $\mathrm{firstop}(t_2) = g$ and
3. $\mathrm{firstop}(t_1) = f$ and $\mathrm{firstop}(t_2) = g$.

Case 3: $\mathrm{op}(t_1) > 1$ and $\mathrm{op}(t_2) > 1$

Theorem

Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n), $\mathrm{op}(t_1) > 1$, $\mathrm{op}(t_2) > 1$, $t_1 = f(s_1, \ldots, s_m)$ and $t_2 = f(s'_1, \ldots, s'_m)$ where $s_1, \ldots, s_m, s'_1, \ldots, s'_m \in W_{(m,n)}(X)$. Then the following conditions are equivalent:

(1) σ_{t_1,t_2} is idempotent;

(2) if $x_j \in \mathrm{var}(t_1)$ where $1 \leq j \leq m$, then $s_j = x_j$ and $t_2 = S^m(t_1, k^1_1, \ldots, k^1_m)$ where $k^1_j \in X$ or $k^1_j = S^m(t_1, k^2_1, \ldots, k^2_m)$ where $k^2_j \in X$ or $k^2_j = S^m(t_1, k^3_1, \ldots, k^3_m) \ldots$ where $k^{l-1}_j \in X$ or $k^{l-1}_j = S^m(t_1, k^l_1, \ldots, k^l_m)$ where $k^l_j \in X$ for some $l \in \mathbb{N}$.

σ_{t_1,t_2}

1. $t_1 = f(g(x_3, x_2), x_2, x_3)$
   $t_2 = f(g(x_1, x_2), x_2, x_1) = S^3(t_1, \boxed{\text{any}}, x_2, x_1)$
2. $t_1 = f(g(x_3, x_2), x_2, x_3)$
   $t_2 = f(g(x_1, x_1), x_1, x_1) = S^3(f(g(x_1, x_1), x_3, x_3), x_1, x_2, x_1)$

Case 3: $\mathrm{op}(t_1) > 1$ and $\mathrm{op}(t_2) > 1$

Theorem

Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m, n), $\mathrm{op}(t_1) > 1$, $\mathrm{op}(t_2) > 1$, $t_1 = f(s_1, \ldots, s_m)$ and $t_2 = g(s'_1, \ldots, s'_n)$ where $s_1, \ldots, s_m, s'_1, \ldots, s'_n \in W_{(m,n)}(X)$. Then the following conditions are equivalent:

(1) σ_{t_1,t_2} is idempotent.

(2) the following statements hold:

  (2.1) if $x_j \in \mathrm{var}(t_1)$ where $1 \leq j \leq m$, then $s_j = x_j$;

  (2.2) if $x_k \in \mathrm{var}(t_2)$ where $1 \leq k \leq n$, then $s'_k = x_k$.

σ_{t_1,t_2}

1. $t_1 = f(x_1, f(x_3, x_1, x_4), x_3)$
   $t_2 = g(x_1, f(x_3, x_1, x_4))$
2. $t_1 = f(\boxed{x_2}, f(x_3, x_1, x_4), x_3)$
   $t_2 = g(x_1, f(x_3, x_1, \boxed{x_2}))$

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