Varieties with Property Going Up

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Introduction

Study of Properties **Going Up** and **Lying Over**:

- originated in ring theory;
- studied for field extensions;
- studied in class field theory;
- more recently, studied in MV–algebras and abelian lattice–orderred groups.

- We have initiated the study of these properties in the general setting of **congruence–modular algebras**.
- This is a sequel of the research I have presented at the **AAA91**, in Brno.
- Here we point out some types of equational classes whose every member fulfills these properties, thus strenghthening some results in the first part of this work, such as the fact that, in the class of Boolean algebras and that of residuated lattices, all morphisms are admissible and fulfill properties Going Up and Lying Over.

Some Notations and Definitions

- the direct product of a family of functions: the usual componentwise definition
- M and N: sets
- $f : M \rightarrow N$
- $f^2 = f \times f : M^2 \rightarrow N^2$
- the direct image of f^2: for any $X \subseteq M^2$, $f(X) = f^2(X) = \{(f(a), f(b)) \mid (a, b) \in X\}$
- the inverse image of f^2: for any $Y \subseteq N^2$, $f^*(Y) = (f^2)^{-1}(Y) = \{(a, b) \in M^2 \mid (f(a), f(b)) \in Y\}$

- any algebra shall be designated by its support set
- any algebra shall be considerred non–empty
- *trivial algebra*: one–element algebra
- *non–trivial algebra*: algebra with at least two distinct elements

- abelian lattice–orderred groups, MV–algebras, residuated lattices: equational classes of algebras related to non–classical logics

Lattices of Congruences

- A : an algebra
- $(\mathrm{Con}(A), \vee, \cap, \Delta_A, \nabla_A)$: the bounded lattice of congruences of A: a complete lattice
- A : *congruence–modular* iff the lattice $\mathrm{Con}(A)$ is modular
- A : *congruence–distributive* iff the lattice $\mathrm{Con}(A)$ is distributive

- for all $\theta \in \mathrm{Con}(A)$, $[\theta) = \{\alpha \in \mathrm{Con}(A) \mid \theta \subseteq \alpha\}$

- for all $X \subseteq A^2$: $Cg_A(X) =$ the congruence of A generated by X
- for all $a, b \in A$: $Cg_A(a, b) = Cg_A(\{(a, b)\})$: the principal congruence of A generated by (a, b)

Morphisms and Congruences in Classes of Algebras

- $\mathcal{C}$: a class of algebras of the same type

- A and B: algebras in $\mathcal{C}$
- $f : A \to B$: a morphism in $\mathcal{C}$
- for all $\alpha \in \mathrm{Con}(A)$, $f(\alpha) \in \mathrm{Con}(f(A))$
- for all $\beta \in \mathrm{Con}(B)$, $f^*(\beta) \in \mathrm{Con}(A)$
- the *kernel* of f: $\mathrm{Ker}(f) = f^*(\Delta_B) \in \mathrm{Con}(A)$

- $\mathcal{C}$: *congruence–modular* iff each algebra in $\mathcal{C}$ is congruence–modular; example: the class of commutative unitary rings
- $\mathcal{C}$: *congruence–distributive* iff each algebra in $\mathcal{C}$ is congruence–distributive; examples: the classes of lattices, MV–algebras, residuated lattices
- $\mathcal{C}$: *semi–degenerate* iff no non–trivial algebra in $\mathcal{C}$ has trivial subalgebras, or, equivalently, iff, for all algebras A in $\mathcal{C}$, $\nabla_A = A^2$ is a finitely generated congruence; examples: the classes of unitary rings, bounded lattices, MV–algebras, residuated lattices

The Commutator, and Prime Congruences

- $\mathcal{C}$: a congruence–modular equational class of algebras of the same type
- hence in $\mathcal{C}$ there exists the *commutator*: for every algebra A in $\mathcal{C}$, $[\cdot,\cdot]_A : \mathrm{Con}(A) \times \mathrm{Con}(A) \to \mathrm{Con}(A)$, such that, for all $\alpha, \beta \in \mathrm{Con}(A)$, $[\alpha,\beta]_A$ is the least of the congruences $\mu \in \mathrm{Con}(A)$ with these properties:
  1. $\mu \subseteq \alpha \cap \beta$
  2. for any algebra B from $\mathcal{C}$ and any surjective morphism in $\mathcal{C}$ $f : A \to B$, $\mu \vee \mathrm{Ker}(f) = f^*([f(\alpha \vee \mathrm{Ker}(f)), f(\beta \vee \mathrm{Ker}(f))]_B)$
- we recall that the commutator is unique (for any congruence–modular equational class, in each of its members)
- if $\mathcal{C}$ is congruence–distributive, then, in every member A of $\mathcal{C}$, the commutator, $[\cdot,\cdot]_A$, equals the intersection of congruences
- A: an algebra in $\mathcal{C}$
- $\phi \in \mathrm{Con}(A) \setminus \{\nabla_A\}$
- ϕ is called a *prime congruence* iff, for all $\alpha, \beta \in \mathrm{Con}(A)$, $[\alpha,\beta]_A \subseteq \phi$ implies $\alpha \subseteq \phi$ or $\beta \subseteq \phi$
- $\mathrm{Spec}(A)$ = the set of the prime congruences of A

Definitions for Admissibility, GU and LO

From now on:

- $\mathcal{C}$: a semi–degenerate congruence–modular equational class of algebras of the same type
- A and B: algebras in $\mathcal{C}$
- $f : A \to B$: a morphism in $\mathcal{C}$

Definition

- We call f an *admissible morphism* iff, for all $\psi \in \mathrm{Spec}(B)$, we have $f^*(\psi) \in \mathrm{Spec}(A)$ (that is $f^*(\mathrm{Spec}(B)) \subseteq \mathrm{Spec}(A)$).

Now assume that f is admissible.

- We say that f fulfills property *Going Up* (abbreviated *GU*) iff, for all $\psi \in \mathrm{Spec}(B)$:

$$[f^*(\psi)) \cap \mathrm{Spec}(A) \subseteq f^*([\psi) \cap \mathrm{Spec}(B))$$

- We say that f fulfills property *Lying Over* (abbreviated *LO*) iff:

$$[\mathrm{Ker}(f)) \cap \mathrm{Spec}(A) \subseteq f^*(\mathrm{Spec}(B))$$

They Are Non–trivial; GU Implies LO

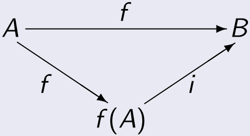
- GU $\Rightarrow$ LO
- surjectivity $\overset{\Rightarrow}{\not\Leftarrow}$ admissibility
- not all morphisms are admissible
- admissibility $\not\Rightarrow$ LO, thus admissibility $\not\Rightarrow$ GU

I have given negative examples for the implications above in the class of bounded lattices.

Study of GU and LO Reduces to Embeddings

Proposition

Let $i : f(A) \to B$ be the canonical embedding.



Then the following hold:

- f is admissible iff i is admissible;
- if f and i are admissible, then: f fulfills GU iff i fulfills GU;
- if f and i are admissible, then: f fulfills LO iff i fulfills LO.

Preservation Properties, Topological Characterization

Admissibility, GU and LO are **preserved by**:

- composition
- quotients
- finite direct products, if $\mathcal{C}$ has no skew congruences, for instance if $\mathcal{C}$ is congruence–distributive, since congruence–distributive varieties have no skew congruences

A **topological characterization for GU**:

- f is admissible, then: f fulfills GU iff the restriction $f^* \mid_{\mathrm{Spec}(B)} : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a closed map with respect to the Stone topologies

EDPC and Admissibility $\Rightarrow$ GU

From now on:

- τ: a type of universal algebras
- $\mathcal{V}$: an equational class of τ–algebras
- for any term t over τ and any algebra M from $\mathcal{V}$, t^M shall be the derivative operation of M associated to t

- $\mathcal{V}$ is called a *variety with equationally definable principal congruences* (abbreviated, *EDPC*) iff there exist an $n \in \mathbb{N}^*$ and terms over τ $p_1, \ldots, p_n, q_1, \ldots, q_n$ of arity 4, such that, for any algebra M from $\mathcal{V}$ and any $a, b, c, d \in M$: $(c, d) \in Cg_M(a, b)$ iff, for all $i \in \overline{1, n}$, $p_i^M(a, b, c, d) = q_i^M(a, b, c, d)$.
- It is known that varieties with EDPC are congruence–distributive.
- See many examples of varieties with EDPC in the papers of Blok and Pigozzi.

We have proved that:

- if $\mathcal{C}$ has EDPC, then all admissible morphisms in $\mathcal{C}$ fulfill GU, thus also LO.

Discriminator Varieties Fulfill GU

- $\mathcal{V}$ is called a *discriminator variety* iff there exists a term t over τ of arity 3 such that, for every subdirectly irreducible algebra M in $\mathcal{V}$ and all $a, b, c \in M$: $t^M(a, b, c) = \begin{cases} a, & \text{if } a \neq b \\ c, & \text{if } a = b \end{cases}$
- It is known that each discriminator variety has EDPC.
- It is known that discriminator varieties include Boolean algebras, Post algebras, n–valued MV–algebras, monadic algebras, cylindric algebras, Gödel residuated lattices.

We have proved that:

- if $\mathcal{C}$ is a discriminator variety, then all morphisms in $\mathcal{C}$ are admissible, hence, by the result above on varieties with EDPC, all morphisms in $\mathcal{C}$ fulfill GU, thus also LO.

Congruence–distributive Varieties with PIP Fulfill GU

- We say that $\mathcal{V}$ has the *principal intersection property* (abbreviated, *PIP*) iff, for any algebra M in $\mathcal{V}$, the intersection of any two principal congruences of M is a principal congruence of M.
- It is known that any congruence–distributive variety with PIP has EDPC.
- The variety of distributive lattices and that of residuated lattices are congruence–distributive and have the PIP.

We have shown that:

- if $\mathcal{C}$ is congruence–distributive and has the PIP, then all morphisms in $\mathcal{C}$ are admissible, hence, by the result above on varieties with EDPC, all morphisms in $\mathcal{C}$ fulfill GU, thus also LO;
- this includes the variety of *bounded* distributive lattices (remember that $\mathcal{C}$ is semi–degenerate) and that of residuated lattices.

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THANK YOU FOR YOUR ATTENTION!