On algebras with a linear bound on the length of terms

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Terms

Definition

Let **A** be an algebra on the language $\mathcal{L}$. For each $n \in \mathbb{N}$ and variables $x_1, \ldots, x_n$ we define the set of all terms with variables $x_1, \ldots, x_n$ in abbreviation $T(x_1, \ldots, x_n)$ as the smallest set with

1. $x_i \in T(x_1, \ldots, x_n)$ for each $i \in \{1, \ldots, n\}$;
2. if $t_1, \ldots, t_k \in T(x_1, \ldots, x_n)$ and $f \in \mathcal{L}$ of the arity $k \in \mathbb{N}$ then $f(t_1, \ldots, t_k) \in T(x_1, \ldots, x_n)$.

Example

In the group $(\mathbb{Z}, +)$ we have $(x + y) + z \in T(x, y, z)$.

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Term operations

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Let $n \in \mathbb{N}$. With each term $t(x_1, \ldots, x_n)$ in the language of the algebra $\mathbf{A}$ we associate an n-ary term operation by interpretation of each operation symbol with corresponding operation in $\mathbf{A}$. The set of all n-ary term functions of $\mathbf{A}$ we denote by $\mathrm{Clo}_n(\mathbf{A})$.

Remark

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Length of terms

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Let $n \in \mathbb{N}$. The length of terms is a function $|| \cdot || : T(x_1, \dots, x_n) \to \mathbb{N}$ such that:

1. $||x_1|| = \cdots = ||x_n|| = 1$;
2. if $k \in \mathbb{N}$, $f \in \mathcal{L}$ and $t_1, \dots, t_k \in T(x_1, \dots, x_n)$ then $||f(t_1, \dots, t_k)|| = 1 + ||t_1|| + \cdots + ||t_k||$.

Example

$||(x+y)+z|| = 1+||x+y||+||z|| = 1+(1+||x||+||y||)+||z|| = 5.$

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Term operations versus terms

Number of functions in finite algebras

If $\mathbf{A}$ is a finite algebra and $n \in \mathbb{N}$ then there is a finite number of distinct n-ary functions on the underlying set.

Remark

For each $n \in \mathbb{N}$, $T(x_1, \ldots, x_n)$ is an infinite set.

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Infinitely many different terms in $T(x_1, \ldots, x_n)$ represent the same n-ary term function, but we need only finitely many of them to represent all distinct n-ary term functions of the given algebra $\mathbf{A}$.

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Circuit complexity problem

How to check?

Let $n \in \mathbb{N}$. Give an n-ary function on a finite algebra. Is it a term function (a circuit)?

When to stop?

A computer program can check all the n-ary terms starting from the smallest length, but when to stop? We should know the minimal length of terms such that all distinct term functions can be represented by terms of the length at most n.

The minimal length of terms

$$\gamma_{\mathbf{A}}(n) := \min\{m \in \mathbb{N} \mid (\forall f \in \mathrm{Clo}_n \mathbf{A})(\exists \mathtt{t})(||\mathtt{t}|| \leq m \wedge \mathtt{t}^{\mathbf{A}} = f)\}$$

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General bounds

Theorem (E. Aichinger and N. M., 2016., Upper bound)

Let $\mathbf{A} = (A, F)$ be an algebra, let $n \in \mathbb{N}$ and let $m = max\{ar(f) \mid f \in F\}$. Then $\gamma_{\mathbf{A}}(n) \leq 1 + m + \cdots + m^{|\mathrm{Clo}_n \mathbf{A}|-1}$.

Proposition (Lower bound)

Let $\mathbf{A} = (A, F)$ be an algebra, let $n \in \mathbb{N}$ and let $|F| = m$, $m \in \mathbb{N}$. If $\mathbb{F}_{\mathcal{V}(\mathbf{A})}(n)$ is the free algebra in the variety $\mathcal{V}(\mathbf{A})$ generated by $\mathbf{A}$ then, $\gamma_{\mathbf{A}}(n) \geq \log_{(m+n+2)} |\mathbf{F}_{\mathcal{V}(\mathbf{A})}(n)|$.

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Functionally complete algebras

Definition

Functionally complete algebra is an algebra such that each function is a polynomial function.

Theorem (E. Aichinger and N.M., 2015)

Let $\mathbf{A} = (A, F)$ be a functionally complete algebra and let $n \in \mathbb{N}$. Then there is a $c \in \mathbb{N}$ such that $\gamma_{\mathbf{A}}(n) \leq |A|^{cn}$.

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Around groups

Theorem (G. Horváth and Ch. Nehaniv, 2014)

Let $n, k \in \mathbb{N}$ and $\mathbf{G}$ be a finite k-nilpotent group. Then $\gamma_{\mathbf{G}}(n) \leq c \cdot n^k$, where c is a constant that depends on $\mathbf{G}$.

„Easy" expanded groups

Easy Ω-groups

Let $\mathbf{V} = (V, +, -, 0, f)$ be a finite group $(V, +, -, 0)$ with one additional unary operation $f : V \to V$ such that $f(0) = 0$.

Easy 3-supernilpoitent expanded group

Let $\mathbf{V} = (V, +, -, 0, f)$ be a finite 3-supernilpotent expanded group, where f is a unary function such that $f(0) = 0$.

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Commutators and higher commutators

In Groups

If H, K are normal subgroups of a group **G** then $[H, K]$ is a normal subgroup generated by $\{[h,k] \mid h \in H,\ k \in K\}$, where $[h,k] := h^{-1}k^{-1}hk$ for all $h \in H$ and $k \in K$.

A. Bulatov, 2001

The term condition n-ary commutator $[\underbrace{\bullet, \ldots, \bullet}_{n}]$ in a Mal'cev algebra is an n-ary operation on Con **A**.

Definition (3-supernilpotent)

Mal'cev algebras that satisfy $[1,1,1,1] = 0$ are called 3-supernilpotent.

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The bound

Theorem (E. Aichinger, M. Lazić and N. M., 2014)

Let $\mathbf{V} = (V, +, -, 0, f)$ be a finite 3-supernilpotent expanded group, where f is a unary function such that $f(0) = 0$ and let $n \in \mathbb{N}$. Then $\gamma_{\mathbf{V}}(n) \leq an^3 + bn^2 + cn + d$, where
$a = (F+1)^3(24EF^3 + 90EF^2 + 107EF + 4F^2 + 5F + 50E + 2)$,
$b = \frac{1}{2}(F+1)^2(9EF^2 + 27EF + 2F + 18E + 2)$,
$c = \frac{1}{2}(F+1)(EF + 2E + 2)$ and $d = -(F+1)$.

Constants

In $\mathbf{V}$ the group $(V, +, -, 0)$ has a finite exponent $expV$ because V is a finite set. We denote $E = expV - 1$.
Since V is a finite set there are $F \in \mathbb{N}$ and $k \leq F$ such that $f^{F+1} = f^k$. We choose the smallest such F.

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The Goal

The task

Find a criteria to decide whether the bound is polynomial?

Conjecture

In case of Mal'cev algebras the bound is polynomial if and only if the algebra is supernilpotent.

Conjecture for linear case

In case of Mal'cev algebras the bound is linear if and only if the algebra is abelian.

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Abelian Mal'cev algebras with prime number of elements

Remark

If we take abelian algebra $(\mathbb{Z}_p, F)$ for a prime number p, then for each n-ary $f \in F$ there are $a_1, \ldots, a_n, b \in \mathbb{Z}_p$ such that $f(x_1, \ldots, x_n) = a_1x_1 + \cdots + a_nx_n + b$ for all $x_1, \ldots, x_n \in \mathbb{Z}_p$.

A Plan

We analyze the case: $b = 0$ and $|F| = 1$.

Proposition (Special case of Horvat and Nehaniv Result)

Let $p \in \mathbb{N}$ such that p is a prime. Then there exist a linear polynomial $f(n)$ such that $\gamma_{\mathbb{Z}_p}(n) \leq f(n)$ for all $n \in \mathbb{N}$.

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Term equivalences

Theorem (E. Aichinger and N. M., 2016)

Let p be a prime and $a, b \in \{1, \ldots, p-1\}$ such that $a + b \not\equiv 1 \pmod{p}$. Then $\mathbf{A} = (\mathbb{Z}_p, ax + by)$ is term equivalent to $(\mathbb{Z}_p, +)$.

Corollary (E. Aichinger and N. M., 2016)

Let p be a prime, let $k \geq 2$, let $\alpha_1, \ldots, \alpha_k \in \{1, \ldots, p-1\}$ with $\alpha_1 + \ldots + \alpha_k \not\equiv 1 \pmod{p}$ and let $g(x_1, \ldots, x_k) = \alpha_1 x_1 + \ldots + \alpha_k x_k$ for all $x_1, \ldots, x_k \in \mathbb{Z}_p$. Then $\mathbf{A} = (\mathbb{Z}_p, g)$ is term equivalent to $(\mathbb{Z}_p, +)$.

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Nonidempotent case

Lemma (E. Aichinger and N. M., 2016)

Let p be a prime and $a, b \in \{1, \dots, p-1\}$. Then $(\mathbb{Z}_p, ax + by)$ is term equivalent to $(\mathbb{Z}_p, +)$ if and only if αx is a term operation of $(\mathbb{Z}_p, ax + by)$ for all $\alpha \in \{0, \dots, p-1\}$.

Theorem (E. Aichinger and N. M., 2016)

Let p be a prime, let $k \geq 2$, let $\theta_1, \dots, \theta_k \in \{1, \dots, p-1\}$ with $\theta_1 + \dots + \theta_k \not\equiv 1 \pmod{p}$, let $u(x_1, \dots, x_k) = \theta_1 x_1 + \dots + \theta_k x_k$ for all $x_1, \dots, x_k \in \mathbb{Z}_p$ and let $\mathbf{A} = (\mathbb{Z}_p, u)$. Then there is a linear polynomial $f(n)$ such that $\gamma_{\mathbf{A}}(n) \leq f(n)$.

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Idempotent case

Lemma (E. Aichinger and N. M., 2016)

Let p be a prime, let $g \in \mathrm{Clo}(\mathbb{Z}_p, +)$ be idempotent, let $\mathbf{A} = (\mathbb{Z}_p, g)$ and let $n \in \mathbb{N}$. Then an n-ary function w on A is a term function of $\mathbf{A}$ if and only if $w(x_1, \ldots, x_n) = \alpha_1 x_1 + \ldots + \alpha_n x_n$ for all $x_1, \ldots, x_n \in \mathbb{Z}_p$, where $\alpha_1, \ldots, \alpha_n$ are constants from the set $\{1, \ldots, p-1\}$ such that $\alpha_1 + \ldots + \alpha_n \equiv 1 \pmod{p}$.

Theorem (E. Aichinger and N. M., 2016)

Let p be a prime, let $f \in \mathrm{Clo}(\mathbb{Z}_p, +)$ and let $\mathbf{A} = (\mathbb{Z}_p, f)$. If f is idempotent then there is a linear polynomial $q(n)$ such that $\gamma_{\mathbf{A}}(n) \leq q(n)$ for all $n \in \mathbb{N}$.

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Main theorem

Theorem (E. Aichinger and N. M., 2016)

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Thank You for the Attention!