Duality for dyadic intervals

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Duality

Let $\mathcal{A}$ and $\mathcal{X}$ be categories. We say that there is a *dual equivalence* or simply *duality* between $\mathcal{A}$ and $\mathcal{X}$ if there are contravariant functors

$$D : \mathcal{A} \to \mathcal{X} \text{ and } E : \mathcal{X} \to \mathcal{A}$$

such that both $DE = E \circ D$ and $ED = D \circ E$ are naturally isomorphic with the corresponding identity functors on $\mathcal{A}$ and $\mathcal{X}$ respectively.

In many cases the functors of the duality are represented by a *schizophrenic object*. The schizophrenic object T appears simultaneously as an object $\underline{\mathrm{T}}$ of $\mathcal{A}$ and as an object $\underset{\sim}{\mathrm{T}}$ in $\mathcal{X}$. The underlying sets of $\underline{\mathrm{T}}$ and $\underset{\sim}{\mathrm{T}}$ coincide (with T).

The functors D and E are defined on objects and morphisms by

$$
\begin{array}{ccccc}
A & & \mathcal{A}(A, \underline{\mathrm{T}}) & & fx : A \to B \to \underline{\mathrm{T}} \\
\downarrow f & \overset{D}{\mapsto} & \uparrow f^D & & \uparrow \\
B & & \mathcal{A}(B, \underline{\mathrm{T}}) & & x : B \to \underline{\mathrm{T}}.
\end{array}
$$

$$
\begin{array}{ccccc}
X & & \mathcal{X}(X, \underset{\sim}{\mathrm{T}}) & & \varphi\alpha : X \to Y \to \underset{\sim}{\mathrm{T}} \\
\downarrow \varphi & \overset{E}{\mapsto} & \uparrow \varphi^E & & \uparrow \\
Y & & \mathcal{X}(Y, \underset{\sim}{\mathrm{T}}) & & \alpha : Y \to \underset{\sim}{\mathrm{T}}
\end{array}
$$

Duality

Theorem (J. D. H. Smith, A. Romanowska)

A τ-algebra A is entropic iff for each τ-algebra X, the morphism set $\underline{\underline{\tau}}(X, A)$ is a subalgebra of the power τ-algebra A^X.

Corollary

If $\mathcal{K}$ is a prevariety of entropic algebras, then for each pair A, B of $\mathcal{K}$-algebras, the morphism set $\mathcal{K}(B, A)$ is again a $\mathcal{K}$-algebra.

Duality

Romanowska, Ślusarski and Smith described a duality between the category of (real) polytopes (finitely generated real convex sets considered as barycentric algebras) and a certain category of intersections of hypercubes, considered as barycentric algebras with additional constant operations. The duality is given by a schizophrenic object, the unit real interval $I = [0, 1]$. The duality for real intervals is trivial. All real intervals are isomorphic, and the dual of any interval is the square $I \times I$.

Convex sets

Let $\mathbb{D}$ be the faithfull affine spaces over the principal ideal domain $\mathbb{D} = \mathbb{Z}[1/2] = \{m/2^n \mid m, n \in \mathbb{Z}\}$ of dyadic rational numbers.

Definition

Let A be a faithful affine $\mathbb{D}$-space. For $x, y \in A$, let $x \circ y = (x+y)/2 = xy\underline{1/2}$ be the arithmetical mean of x and y. Then the subreduct $(B, \circ)$ of the reduct $(A, \circ)$ is called an *algebraic dyadic convex sets*.

Definition

A subset of $\mathbb{D}^k$, for $k = 1, 2, \ldots$, is called a *geometric dyadic convex set* or briefly just a *dyadic convex set*, if it is the intersection of a convex subset C of $\mathbb{R}^k$ with its subspace $\mathbb{D}^k$.

Dyadic intervals

Definition

By an interval of $\mathbb{D}$ we mean a subset $[a, b] := \{x \in \mathbb{D} \mid a \leq x \leq b\}$, for $a, b \in \mathbb{D}$. In particular, $\mathbb{D}_1$ denotes the dyadic unit interval, the intersection $I \cap \mathbb{D}$ of the unit real interval $I = [0, 1]$ and the dyadic line $\mathbb{D}$.

Dyadic intervals are considered as commutative binary modes.

Dyadic intervals

Theorem [K. Matczak, A. Romanowska, J. D. H. Smith]

Each non-trivial interval of $\mathbb{D}$ is isomorphic to some interval $[0, k]$, where k is an odd positive integer. Two such intervals are isomorphic precisely when their right hand ends are equal.

If an interval of $\mathbb{D}$ is isomorphic to some interval $[0, k]$, where k is an odd positive integer, then we say that it is of *type* k and denote by $\mathbb{D}_k$.
Each dyadic interval of type $k > 1$ is 3-*generated*, that means it is minimally generated by three elements.

Dyadic intervals

Theorem

Each interval $[0, k]$, where k is an odd positive integer is generated by any of the following sets: $\{0, 1, k\}$, $\{0, g, k\}$, $\{0, 2^n, k\}$ where $gcd(g, k) = 1$ and 2^n it the greatest power of two not greater than k.

For example:

$$< 0, 1, 5 > \cong < 0, 3, 5 > \cong < 0, 4, 5 > \cong < 0, \frac{1}{2}, 5 > \cong \mathbb{D}_5,$$

and

$$< 0, 3 > \cong < 0, 5 > \cong < 0, 1 > \cong \mathbb{D}_1.$$

Moreover

$$< 0, 5 > \leq \mathbb{D}_5.$$

The category $\mathcal{A}$

The category $\mathcal{A}$ we are interested in will be the category $\mathcal{DJ}$ of commutative binary modes isomorphic to dyadic intervals. This is a subcategory of the category $\mathcal{Q} = \mathsf{Q}(\mathbb{D})$ of algebraic dyadic convex sets, which is a subquasivariety of the variety of commutative binary modes. Morphisms of the category $\mathcal{DJ}$ are relative groupoid homomorphisms, that means homomorphisms from members of $\mathcal{Q}$ into members of $\mathcal{Q}$. The unit interval $\mathbb{D}_1$ will play a role of a schizophrenic object. For an interval J in $\mathcal{DJ}$, the representation space X will be constructed on the set $\mathcal{Q}(J, \mathbb{D}_1)$ of homomorhisms from J to $\mathbb{D}_1$.

Homomorphic images of dyadic intervals

Lemma

Let h be a homomorphism of a non-trivial dyadic interval J of type k into $\mathbb{D}_1$. Then the homomorphic image $h(J)$ is isomorphic to J, and hence it is also of type k, or else it is trivial.

Note as well that $h(\mathbb{D}_k)$ is not necessarily an interval (But it is always isomorphic with an interval.). However, it is always determined by the end points $h(0)$ and $h(k)$, and moreover $0 \leq h(0), h(k) \leq 1$. Consequently, we can identify the elements of $\mathcal{Q}(\mathbb{D}_k, \mathbb{D}_1)$ with the pairs $(h(0), h(k))$.

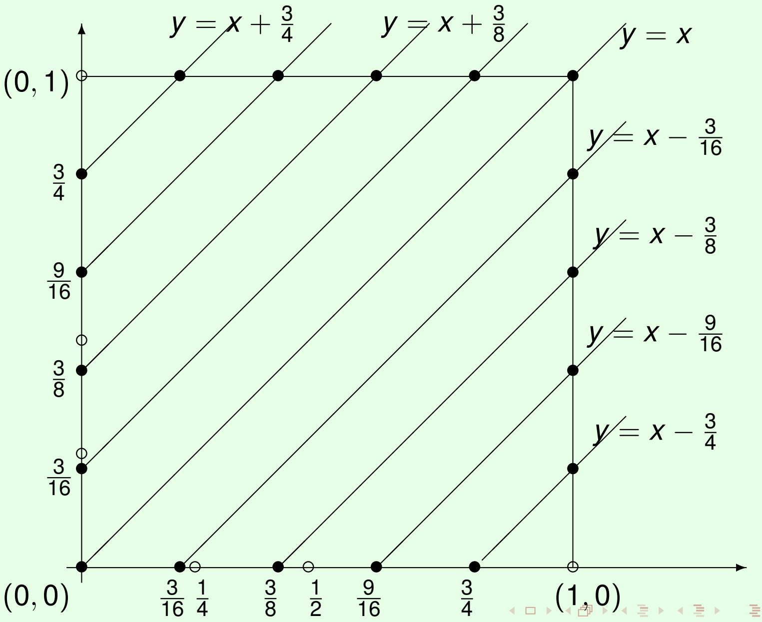
The first dual

Proposition

The groupoid $\mathcal{Q}(\mathbb{D}_k, \mathbb{D}_1)$ is isomorphic to the subgroupoid H_k of the groupoid $\mathbb{D}_1 \times \mathbb{D}_1$ consisting of all points $(a, b) \in \mathbb{D}_1 \times \mathbb{D}_1$ such that k divides the difference $b - a$.

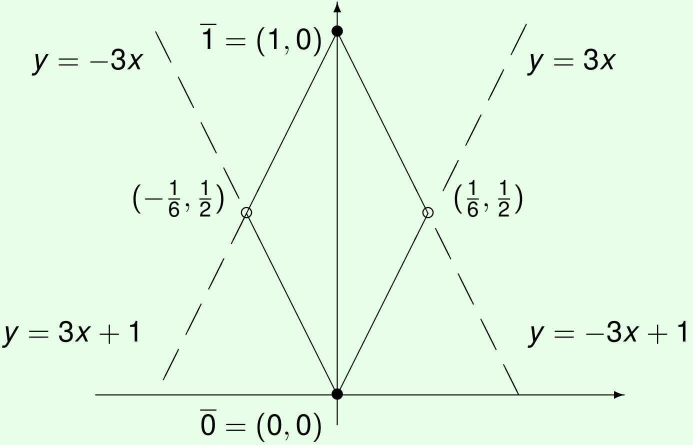
The first dual for the interval $\mathbb{D}_k$ is the groupoid H_k.

The groupoid H_3



The groupoid H'_3

The grouopid H_3 is isomorphic to the groupid H'_3. The groupoid H'_3 has the important property. Every dyadic point of the real rhombus is an element of H'_3.



The first dual

The first dual $X_k := \mathbb{D}_k^D$ of $\mathbb{D}_k$ will be the groupoid H'_k, equipped additionally with constants $\bar{0} = (0,0)$ and $\bar{1} = (0,1)$. Hence X_k is considered as the algebra $(H'_k, \circ, \bar{0}, \bar{1})$. Additionally, we define $\mathbb{D}_0$ to be a trivial interval, and X_0 to be $\mathbb{D}_1$ considered as the groupoid with constants 0 and 1. Note that, for odd $k \geq 3$ the set H'_k is a (dyadic) closed rhombus but without two the two vertices $(-1/(2k), 1/2)$ and $(1/(2k), 1/2)$.
The dual category $\mathcal{X}$ is then described as the category $\widehat{\mathcal{DJ}}$ with the groupoids isomorphic to the groupoids X_k as objects, and with (relative) groupoid homomorphisms respecting the constants as morphisms.

The second dual

To describe the second dual $\mathbb{D}_k^{DE}$ of $\mathbb{D}_k$ we will need a description of the non-trivial proper relative congruences of X_k.

Lemma

The kernel θ_h of a homomorphism $h : (X_k, \circ, \bar{0}, \bar{1}) \to (\mathbb{D}_1, \circ, \bar{0}, \bar{1})$, where $k \geq 1$, is determined by the slope α of a family of parallel lines given by $y = \alpha x + b$ crossing (but not containing) the diagonal $\delta = \{(0, d) \mid d \in \mathbb{D}_1\}$ of X_k. The blocks of θ_h are subgroupoids of H'_k, each consisting of the (dyadic) points of X_k belonging to one line of slope α.

The second dual

Lemma

The set $\widehat{\mathcal{DJ}}(X_k, \mathbb{D}_1)$ *of homomorphisms from* X_k *to* $\mathbb{D}_1$ *forms a commutative binary mode isomorphic to the interval* $\mathbb{D}_k$.

Theorem

There is a duality between the categories $\mathcal{DJ}$ *and* $\widehat{\mathcal{DJ}}$.

Thank You for your attention

Figure : Dual ladybirds