Characterization of Modularity by Means of Cover-Preserving Sublattices

Marcin Łazarz
University of Wrocław

92. Arbeitstagung Allgemeine Algebra
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Sublattices vs cover-preserving sublattices

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Definition. A sublattice K of a lattice L is said to be a **cover-preserving sublattice**, $K \prec L$, iff:

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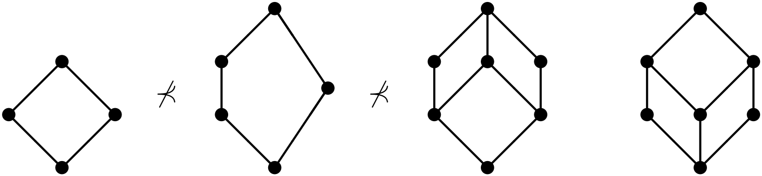

$B_2 \not\prec N_5 \not\prec S_7 \qquad S_7^*$

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Theorem [Ward 1939]. Let L be a finite modular lattice. L is distributive iff there is no cover-preserving sublattice isomorphic to M_3.

Upper continuous and strongly atomic lattices

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Fact.

- Ascending chain condition $\Rightarrow$ (UC)
- Descending chain condition $\Rightarrow$ (SA)

Modularity in upper continuous and strongly atomic lattices

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Theorem [Birkhoff 1933, Crawley–Dilworth 1973].
If lattice L satisfies (UC) and (SA) then the following conditions are equivalent:

1. L is modular,
2. L satisfies (Sm) and (Sm*),

where:

$$(\forall x, y \in L)(x \wedge y \prec x \Rightarrow y \prec x \vee y) \tag{Sm}$$

$$(\forall x, y \in L)(y \prec x \vee y \Rightarrow x \wedge y \prec x) \tag{Sm*}$$

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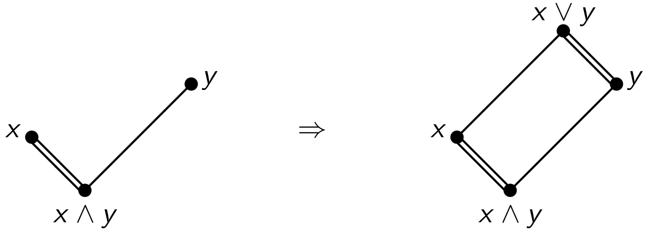
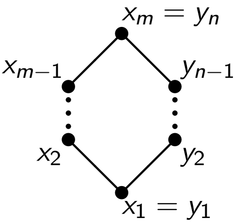


$N_{m,n}$	N_5	S_7	S_7^*
$\neg$(Sm)	$\neg$(Sm)	(Sm)	$\neg$(Sm)
$\neg$(Sm*)	$\neg$(Sm*)	$\neg$(Sm*)	(Sm*)



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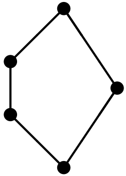
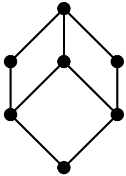
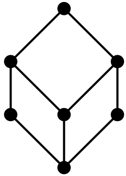


N_5



S_7



S_7^*

$B_2 := N_{3,3} \quad N_5 := N_{4,3}$

J. Jakubík and F. Šik's Characterizations of Modularity

Theorem [Jakubík 1975]. Let L be a lattice of **locally finite length**.[1] Then the following conditions are equivalent:

1. L is modular,
2. $S_7, S_7^*, N_{m,n} \not\leq L$ (for $m \geq 4, n \geq 3$).

[1] Every bounded chain is finite.

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Theorem [Šik 197?]. Let L be a lattice of **locally finite length** satisfying (Sm). Then the following conditions are equivalent:

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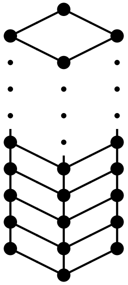

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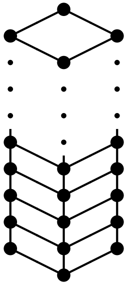

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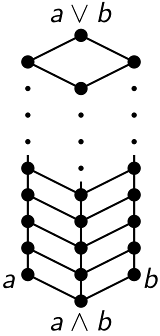

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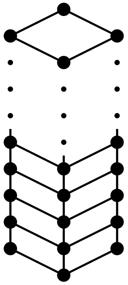

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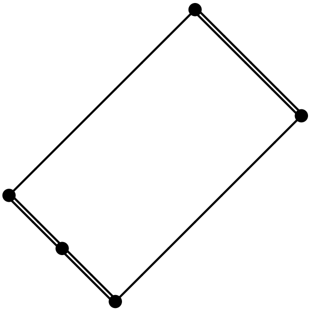

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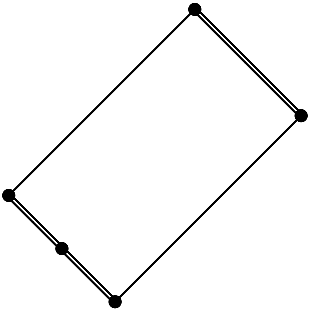

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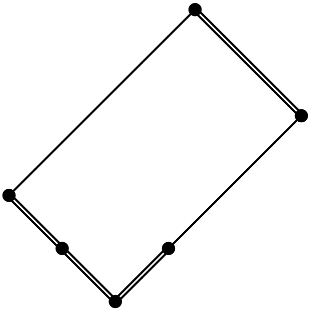

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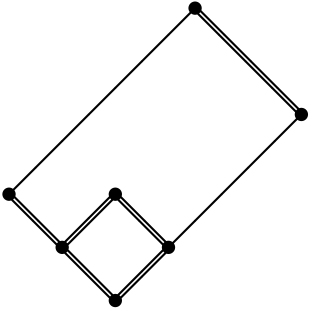

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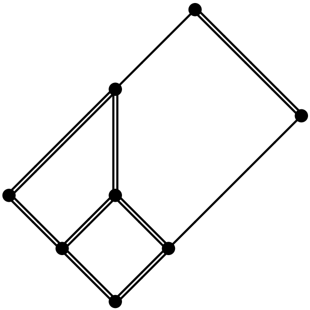

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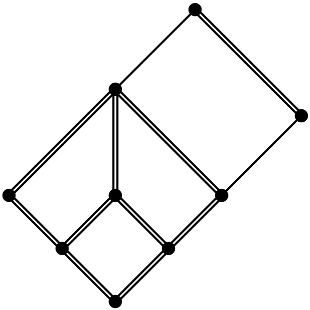

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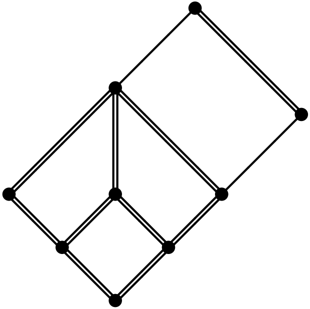

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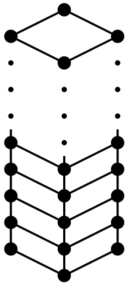

Applications: "finitary" description of modularity...

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Corollary. If L satisfies (UC), (SA), and **(Sm)** then the following conditions are equivalent:

(*i*) L is modular,

(*ii*) every interval of finite length of L is modular.

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Theorem [Łazarz, Siemieńczuk 2015].
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Corollary. If a **modular** lattice L satisfies (UC), (SA) then the following conditions are equivalent:

(i) L is distributive,

(ii) every interval of finite length of L is distributive.

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