# Qualitative Calculi as a generalisation of Tarski's Relation Algebras

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Let $U$ be any set. Consider $\mathcal{P}(U \times U)$ with the following operations:

- union ($\cup$), intersection ($\cap$) and complement ($^-$)
- relational composition ($\circ$) and converse ($^{-1}$)
- identity relation ($Id$), bottom ($\emptyset$), top ($U \times U$)

The structure $\mathfrak{Re}(U) = \langle \mathcal{P}(U \times U); \cup, \cap, \circ, ^-, ^{-1}, Id, \emptyset, U \times U \rangle$ is an algebra of binary relations.

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   - $x \circ (y \cup z) = (x \circ y) \cup (x \circ z)$, $(x \cup y) \circ z = (x \circ z) \cup (y \circ z)$
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5. Or, equivalently, $(x^{-1} \circ (x \circ y)^-) \cup y^- = y^-$.

# Atom structures

An (abstract) relation algebra (RA) is any algebra $\mathbf{A} = \langle A; \vee, \wedge, ;, ^{-}, ^{\smile}, 1', 0, 1 \rangle$ satisfying equations from the previous slide.

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## Definition

Let $X$ be the set of atoms of an atomic RA algebra $\mathbf{A}$. The atom structure $At(\mathbf{A})$ is defined as $At(\mathbf{A}) = (X, E, ^\smile, C)$ where $E$ is the set of atoms below the identity, $^\smile$ is the converse function restricted to atoms, and $C$ is the set of consistent triples of atoms, i.e. those triples of atoms $(a, b, c)$ such that $a \,;\, b \geq c$.

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Atom structures can be given as multiplication tables, e.g.:

| $;$ | $1'$ | $b$ | $a$ |
|---|---|---|---|
| $1'$ | $1'$ | $b$ | $a$ |
| $b$ | $b$ | $1' \vee a$ | $a \vee b$ |
| $a$ | $a$ | $a \vee b$ | $1' \vee b$ |

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*There are non-representable relation algebras.*

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**Theorem (Lyndon, 1950)**

*There are non-representable relation algebras.*

**Theorem (Hirsch and Hodkinson, 2001)**

*Representability for finite RAs is undecidable.*

# Representability by games

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Consider the 16-element Boolean algebra, with atoms $1'$, $b$, $b^{\smile}$, $a$. Define composition on atoms by

| ; | $1'$ | $b$ | $b^{\smile}$ | $a$ |
|---|---|---|---|---|
| $1'$ | $1'$ | $b$ | $b^{\smile}$ | $a$ |
| $b$ | $b$ | $b$ | $1$ | $a \vee b$ |
| $b^{\smile}$ | $b^{\smile}$ | $1$ | $b^{\smile}$ | $a \vee b^{\smile}$ |
| $a$ | $a$ | $a \vee b$ | $a \vee b^{\smile}$ | $1' \vee b \vee b^{\smile}$ |

Let **K** stand for the algebra defined by this table.

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|---|---|---|---|---|
| $=$ | $=$ | $<$ | $>$ | $\#$ |
| $<$ | $<$ | $<$ | $1$ | $<, \#$ |
| $>$ | $>$ | $1$ | $>$ | $>, \#$ |
| $\#$ | $\#$ | $<, \#$ | $>, \#$ | $=, <, >$ |

Let **K** stand for the algebra defined by this table.

**Theorem (McKenzie, 1974)**

**K** *is an abstract relation algebra, but it is not representable.*

# Allen's Interval Algebra

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| | | |
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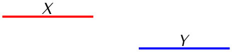
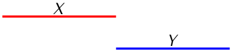
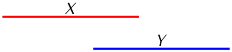
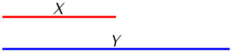
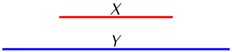
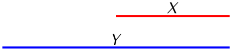
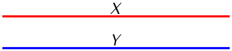
|   | $X$ before $Y$ | $Y$ after $X$ |
|   | $X$ meets $Y$ | $Y$ is met by $X$ |
|   | $X$ overlaps with $Y$ | $Y$ is overlapped by $X$ |
|   | $X$ starts $Y$ | $Y$ is started by $X$ |
|   | $X$ during $Y$ | $Y$ contains $X$ |
|   | $X$ finishes $Y$ | $Y$ is finished by $X$ |
|   | $X$ equals $Y$ | $Y$ equals $X$ |

# Region Connection Calculus (RCC8)

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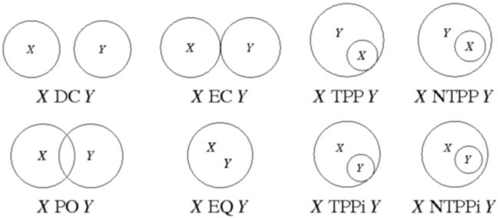


| | | |
|---|---|---|
| *X* DC *Y* | *X* is disconnected from *Y* | *Y* DC *X* |
| *X* EC *Y* | *X* is externally connected to *Y* | *Y* EC *X* |
| *X* TPP *Y* | *X* is a tangential proper part of *X* | *Y* TPPi *X* |
| *X* NTPP *Y* | *X* is a non-tangential proper part of *Y* | *Y* NTPPi *X* |
| *X* PO *Y* | *X* properly overlaps *Y* | *Y* PO *X* |
| *X* EQ *Y* | *X* is equal to *Y* | *Y* EQ *X* |

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## Definition

Let $U$ be a set and let $\Pi$ be a partition of $U \times U$. For partition classes $R$ and $S$ we define their weak composition, by

$$R\,;S = \bigcup\{A \in \Pi \colon (R \circ S) \cap A \neq \emptyset\}.$$

Extend this to arbitrary unions of partition classes, putting

$$\bigcup_{i \in I} R_i\,;\bigcup_{j \in J} S_j = \bigcup_{(i,j) \in I \times J} R_i\,;S_j.$$

# Algebras of relations with weak composition

## Definition

Let $D$ be a set and let $\mathcal{S}$ be a set of binary relations over $D$, that is, $\mathcal{S} \subseteq \mathcal{P}(D \times D)$. We say that $\mathcal{S}$ is a flock if

1. $\mathcal{S}$ forms a boolean set algebra with top element $D \times D$,
2. $Id_D \in \mathcal{S}$,
3. If $A \in \mathcal{S}$ then the converse relation $A^{-1}$ is in $\mathcal{S}$,
4. For all $A, B \in \mathcal{S}$ there is a smallest relation $C \in \mathcal{S}$ containing $A \circ B$.

The last property follows automatically when $\mathcal{S}$ is finite, since it is closed under finite intersections. Indeed, in the finite case, the smallest $C$ containing $A \circ B$ is precisely the weak composition of $A$ and $B$.

# Qualitative representation of a (non-associative) algebra

## Definition

Let $\mathbb{A} = (A, \vee, \wedge, ;, -, \breve{}, 1', 0, 1)$ be an algebra of the signature of relation algebras.

- A qualitative square representation $\phi$ of an algebra $\mathbb{A}$ is an injective mapping of $A$ to the full algebra of binary relations $\mathfrak{Re}(D)$ over some set $D$, such that
  1. $0^\phi = \emptyset$, $1^\phi = D \times D$, $(1')^\phi = Id_D$,
  2. $(a \vee b)^\phi = a^\phi \cup b^\phi$, $(-a)^\phi = (D \times D) \setminus a^\phi$,
  3. $(a^\smile)^\phi = (a^\phi)^\smile$,
  4. $c^\phi \supseteq a^\phi ; b^\phi \iff c \geq a ; b$

  for all $a, b, c \in A$.
- If $(a ; b)^\phi = a^\phi ; b^\phi$ for all $a, b \in \mathbb{A}$ then $\phi$ is a strong square representation, or simply a square representation.

# Representing the non-representable

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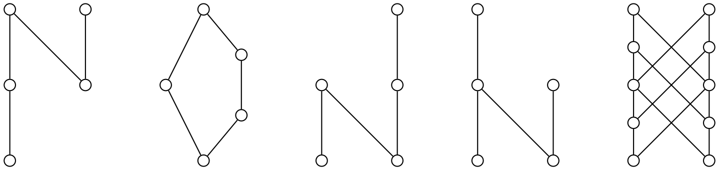

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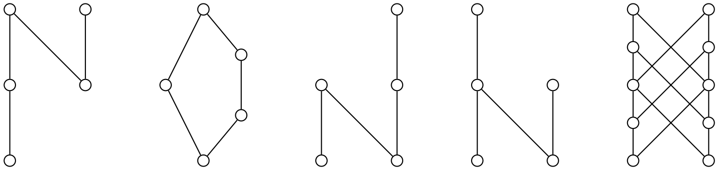

Some representations of **K**.

## Theorem

*The problem of determining whether a finite atom structure has a qualitative repesentation is NP-complete.*

# Networks

## Definition

Let **A** be a non-associative algebra.

- A network $(N, \lambda)$ over **A** consists of a set $N$ of nodes and a function $\lambda : (N \times N) \to \mathbf{A}$.
- A network $(N, \lambda)$ is consistent if
  - (a) $\lambda(x, x) \leq 1'$,
  - (b) $(\lambda(x, y) ; \lambda(y, z)) \wedge \lambda(x, z) \neq 0$, for all nodes $x, y, z \in N$,
  - (c) $\lambda(x, y) \wedge \lambda(y, x)^{\smile} \neq 0$,
  - (d) $\lambda(x, y) \neq 0$, for all nodes $x, y \in N$.
- An atomic network $(N, \lambda)$ is a network where $\lambda(x, y)$ is always an atom of **A**.
- An atomic network is consistent if it is consistent as a network.

# Network satisfiability

- A network $(N, \lambda)$ embeds into a strong representation $\phi$ if there is a map $'$ from $N$ to the base of $\phi$ such that for all $x, y \in N$ we have $(x', y') \in \lambda(x, y)^{\phi}$.

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- Similarly, $(N, \lambda)$ embeds into a qualitative representation $\theta$ if there is a map $'$ from $N$ to the base of a qualitative representation $\theta$ such that for all $x, y \in N$ we have $(x', y') \in \lambda(x, y)^{\theta}$.

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- A network over $\mathbf{A}$ is satisfiable if it embeds into some strong representation of $\mathbf{A}$ and it is qualitatively satisfiable if it embeds into some qualitative representation of $\mathbf{A}$. Clearly, if $(N, \lambda)$ is strongly satisfiable then it is qualitatively satisfiable.

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- A network over **A** is satisfiable if it embeds into some strong representation of **A** and it is qualitatively satisfiable if it embeds into some qualitative representation of **A**. Clearly, if $(N, \lambda)$ is strongly satisfiable then it is qualitatively satisfiable.
- A representation $\phi$ of the finite relation algebra **A** is universal if every consistent atomic network embeds into $\phi$.

# Some (very little) model theory

**Theorem**

*The class of qualitatively representable algebras (QRA) is not strictly elementary. But $V(QRA)$ is what you would expect:*

- $V(QRA) = SP(QRA)$.
- $V(QRA)$ *is a discriminator variety with simple members belonging to QRA.*

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## Theorem

*Let* $\mathbf{A}$ *be a finite algebra of the type of relation algebras, and* $\mathcal{N}_{\mathbf{A}}$ *be the class of consistent atomic networks over* $\mathbf{A}$*.*

1. *If* $\mathcal{N}_{\mathbf{A}}$ *has JEP, then* $\mathbf{A}$ *is a QRA.*
2. *If* $\mathcal{N}_{\mathbf{A}}$ *has JEP and AP, then* $\mathbf{A}$ *is a RRA. Moreover,* $\mathbf{A}$ *has an* $\omega$*-categorical, homogeneous representation.*

# Monotone not-all-equal 3-sat

Given a finite set of clauses $C_1, \ldots, C_n$, such that:

- all literals are positive
- there are precisely three literals in each clause

is there an assignment $v$ such that

- $v$ satisfies $C_1 \wedge \cdots \wedge C_n$
- it is not the case that $v(\ell_1^i) = v(\ell_2^i) = v(\ell_3^i)$, for any clause $C_i = \ell_1^i \vee \ell_2^i \vee \ell_3^i$.

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**Lemma**

*Monotone* NAE-3-SAT *is NP-complete.*

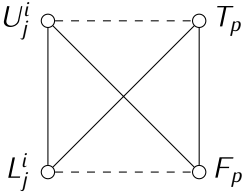
We will now interpret M-NAE-3-SAT in NSP over McKenzie algebra **K**. This will show that network satisfiability for **K** is NP-hard.

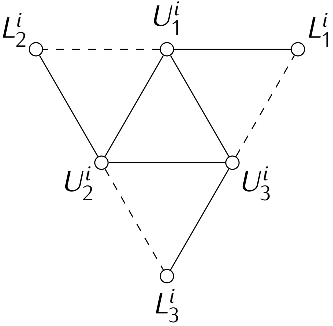
# Interpreting variables, literals and clauses

For an arbitrary instance $I$ of M-NAE-3-SAT we construct a network $(N_I, \lambda_I)$ corresponding to $I$. The vertices of $N_I$ are:

- $T_p$ and $F_p$, for each propositional variable $p$ occurring in $I$,
- $U_1^i, U_2^i, U_3^i, L_1^i, L_2^i, L_3^i$, for literals $\ell_1^i, \ell_2^i, \ell_3^i$ in each clause $C_i$.

The labelling function $\lambda_I$ is required to satisfy conditions illustrated below:



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Next, we have to go the other way: for a satisfiable instance $I$ of M-NAE-3-SAT, we need to find a copy of $(N_I, \lambda_I)$ as a subnetwork of some qualitative representation of $\mathbf{K}$.

# The other way, by example

Consider $v(p) = v(q) = 1$ and $v(r) = v(s) = 0$ on the instance

$$(p \vee q \vee r) \wedge (q \vee r \vee s) \wedge (r \vee s \vee p) \wedge (s \vee p \vee q)$$

where we have literals $\ell_1^1, \ell_3^3, \ell_2^4$ instantiated by $p$, literals $\ell_2^1, \ell_1^2, \ell_3^4$ instantiated by $q$, literals $\ell_3^1, \ell_2^2, \ell_1^3$ instantiated by $r$, and literals $\ell_3^2, \ell_2^3, \ell_1^4$ instantiated by $s$.

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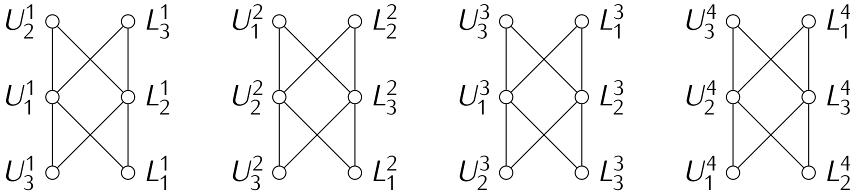

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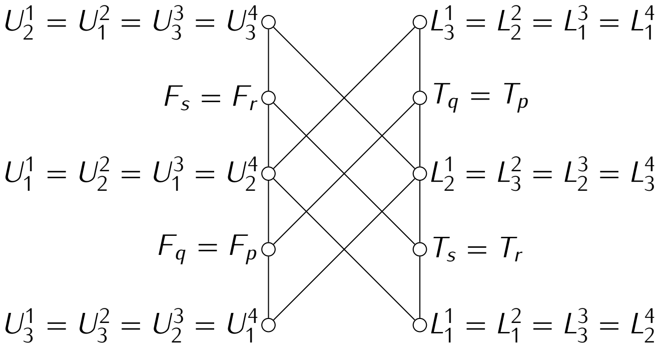

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**Final remark**

*This argument in fact shows NP-hardness of NSPs over certain fragments of any qualitative calculus* $\mathbf{Q}$ *such that:*

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*This subsumes most of the existing results.*

# Thank you!