The radicals of local residuated lattices

AAA92

27-29 May 2016, Prague

Michiro Kondo
Tokyo Denki University, JAPAN

Results

1. For a good residuated lattice X, X is perfect if and only if $X/D(X) \cong \{0,1\}$

2. For a local residuated lattice X, it is relative free of zero divisors if and only if $X = Rad(X) \cup \{\mathbf{0}\}$

3. Every local residuated lattice is strong.

$(X, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ **is called a** ***residuated lattice*** **if**

(1) $(X, \wedge, \vee, 0, 1)$ **is a bounded lattice;**

(2) $(X, \odot, 1)$ **is a monoid;**

(3) $x \odot y \leq z$ **if and only if** $x \leq y \rightarrow z$ **if and only if** $y \leq x \rightsquigarrow z$**.**

For all $x \in X$**, we denote** $x^{-} = x \rightarrow 0$ **and** $x^{\sim} = x \rightsquigarrow 0$**.**

$F \subseteq X$ $(F \neq \emptyset)$ **is called a** *filter* $(F \in Fil(X))$ **if**

(F1) $x, y \in F \Rightarrow x \odot y \in F$

(F2) $x \in F$ **and** $x \leq y \Rightarrow y \in F$

A filter F **is called** *normal* $(F \in Fil_n(X))$ **when**

$x \to y \in F$ **if and only if** $x \rightsquigarrow y \in F$**.**

$$\Downarrow$$

$X/F = (X/F, \wedge, \vee, \odot, \to, \rightsquigarrow, 0/F, 1/F)$ **is a residuated lattice for** $F \in Fil_n(X)$**.**

$F \in Fil(X)$ **(**$F \neq X$**) is called** ***Boolean*** **if**

$$x \vee x^{-}, x \vee x^{\sim} \in F \textbf{ for any } x \in X.$$

Fact **(Rachůnek and Šalounová,2010)**

For $F \in Fil_n(X)$**,**

F **is Boolean** $\Longleftrightarrow$ X/F **is a Boolean algebra.**

By $\mathsf{ord}(x)$**, we mean the least natural number** $n \in \mathbb{N}$ **such that** $x^n = 0$**, where** $x^n = \underbrace{x \odot \cdots \odot x}_{n}$**.** **If no such natural number, then** $\mathsf{ord}(x) = \infty$**.**

We define

$$D(X) = \{x \in X \,|\, \mathsf{ord}(x) = \infty\},$$

the set of all elements of X **with infinite order.**

X **is called** *local* **if** X **contains a unique maximal filter.**

Fact **(Ciungu,2009) The following conditions are equivalent:**

(1) $D(X)$ **is a proper filter of** X**;**

(2) X **is local;**

(3) $D(X)$ **is the unique maximal filter of** X**;**

(4) $\text{ord}(x \odot y) < \infty \Rightarrow \text{ord}(x) < \infty$ **or** $\text{ord}(y) < \infty$.

A local residuated lattice X is called *perfect* **if** $\mathrm{ord}(x) < \infty$ **or (**$\mathrm{ord}(x^-) = \infty$ **and** $\mathrm{ord}(x^\sim) = \infty$**) for every** $x \in X$**.**

Proposition **For a good residuated lattice X,** X **is perfect** $\iff \mathbf{0} \notin [x \vee x^-)$ **and** $\mathbf{0} \notin [x \vee x^\sim)$ $\iff D(X)$ **is a Boolean filter.**

To show that $D(X)$ is normal, we need another topic, *state***.**

A map $s : X \to [0, 1]$ **is called a** *Bosbach state* **if**

(S1) $s(x) + s(x \to y) = s(y) + s(y \to x)$

(S2) $s(x) + s(x \rightsquigarrow y) = s(y) + s(y \rightsquigarrow x)$

(S3) $s(0) = 0$ **and** $s(1) = 1$

Fact **(Kondo,2014) For a Bosbach state s on a residuated lattice X,**

(1) $\ker(s)$ **is a normal filter;**

(2) $X/\ker(s)$ **is an MV-algebra.**

Corollary **For any perfect residuated lattice X, $D(X)$ is a normal filter.**

Considering the quotient algebra $X/D(X)$ by the normal filter $D(X)$,

Theorem For a good residuated lattice X, X is a perfect residuated lattice if and only if $X/D(X)$ is the two-element Boolean algebra, that is,

$$X \text{ is perfect} \iff X/D(X) \cong \{0, 1\}.$$

We define

$Rad(X) = \cap\{M \mid M$ is a maximal filter of $X\}$,

$Rad(X)^* = \{x \in X \mid x \notin Rad(X)\}$,

$x \perp y \iff y^{-\sim} \leq x^-$ **and** $x \oplus y = (y^- \odot x^-)^\sim$.

A map $s : Rad(X)^* \to [0,1]$ **is called a** *local additive measure* **if**

(lam_1**)** $x \perp y$ **and** $x \oplus y \in Rad(X)^*$

$\Rightarrow s(x \oplus y) = s(x) + s(y)$ **for** $x, y \in Rad(X)^*$**;**

(lam_2**)** $s(0) = 0$**.**

X **is called** *relative free of zero divisors* **if**

$\exists y_1, y_2 \in Rad(X)$ **s.t.** $x \odot y_1 = y_2 \odot x = 0$

$\Rightarrow x = 0$

Theorem **A residuated lattice X is relative free of zero divisors if and only if $x^-, x^\sim \in Rad(X)$ implies $x = 0$.**

Corollary **If X is local and relative free of zero divisors, then $D(X) = X - \{0\}$.**

Fact (Extension property) **(Ciungu, Georgescu and Mureşan, 2013) Let X be a local residuated lattice which is relative free of zero divisors. Then every local additive measure s on X can be extended to a 2-valued Riečan state R_s on X.**

$$\Downarrow$$

Theorem

There exists a unique local additive measure on a local residuated lattice which is relative free of zero divisors.

Theorem

Let X be a local residuated lattice relative free of zero divisors and s be a local additive measure on X. Then the extension R_s of s is a Bosbach state if and only if $s(x) = 0$ for $x \in Rad(X)^*$.

A perfect residuated lattice X is called *strong* if $x, y \in Rad(X)^*$ imply $x \vee y \in Rad(X)^*$.

Fact (Ciungu, Georgescu and Mureșan, 2013) Any strong perfect residuated lattice admits a generalized state-morphism.

For a local residuated lattice X, we have $Rad(X) = \{x \in X \mid ord(x) = \infty\}$.

$$\Downarrow$$

Theorem **Every local (and hence perfect) residuated lattice is strong.**

Indeed, for $x, y \in Rad(X)^* = D(X)^*$**, there exist** $m, n \in \mathbb{N}$ **such that** $x^m = y^n = \mathbf{0}$**.**

$\Rightarrow (x \vee y)^{m+n} \leq x^m \vee y^n = \mathbf{0} \vee \mathbf{0} = \mathbf{0}$

$\Rightarrow x \vee y \in D(X)^* = Rad(X)^*$

Combining these results, we conclude that

Theorem **Every local residuated lattice admits a generalized state-morphism.**

Thank you for your attention!!