On rings which are sums of two *PI*-subrings

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Outline

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Let R be an associative ring and R_1, R_2 its subrings. Suppose that for every $r \in R$ there exist $r_1 \in R_1$ and $r_2 \in R_2$ such that $r = r_1 + r_2$. We denote it shortly by

$$R = R_1 + R_2$$

(we keep this notation throughout the talk).

Ex.: $\begin{pmatrix} \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix} + \begin{pmatrix} \mathbb{R} & 0 \\ \mathbb{R} & \mathbb{R} \end{pmatrix}$, where $\mathbb{R}$ is the field of real numbers.

By $A = \mathbb{Z}[x_1, x_2, \ldots, x_n]$ we denote the free ring in indeterminates $x_1, x_2, \ldots, x_n$. A polynomial $f \in A$ is called *monic* if at least one of the monomials of highest degree in the support of f has coefficient 1.

We say that B is a *PI* ring if and only if there is a monic polynomial $f \in A$ such that $\phi(f) = 0$ for every homomorphism $\phi : A \to B$.
Then we say that B satisfies the identity $f = 0$.

Ex.: The ring $M_2(\mathbb{R}) = \left(\begin{smallmatrix} \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} \end{smallmatrix}\right)$ satisfies the identity $[[x_1, x_2]^2, x_3] = (x_1x_2 - x_2x_1)^2 x_3 - x_3(x_1x_2 - x_2x_1)^2 = 0$.

K. I. Beidar and A. V. Mikhalev in the paper:

Generalized polynomial identities and rings which are sums of two subrings, Algebra and Logic 34(1995), 1-5 (English translation).

stated the following problem, which is still open:

- Let $R = R_1 + R_2$. Suppose that R_1 and R_2 satisfy polynomial identities (shortly, are *PI* rings), is then also R a *PI* ring?

The answer to this problem is known to be positive in several cases.

Let $R = R_1 + R_2$.

Kegel (1962)

- If R_1 and R_2 are nilpotent (i.e. they satisfy the identity $x_1 x_2 \cdots x_n = 0$), then so is R.

Puczyłowski-Kępczyk (1996)

- If R_1 and R_2 are nil of bounded index (i.e. they satisfy the identity $x^n = 0$), then so is R.

Let $R = R_1 + R_2$.

Bahturin-Giambruno (1994)

- If R_1 and R_2 are commutative, then R satisfies the identity $[x_1, x_2][x_3, x_4] = 0$.

Beidar-Mikhalev (1995)

- If R_1 satisfies the identity $[x_1, x_2] \cdots [x_{2m-1}, x_{2m}] = 0$ and R_2 satisfies the identity $[x_1, x_2] \cdots [x_{2n-1}, x_{2n}] = 0$ for some $m, n \geqslant 2$, then R is a PI ring.

The proof of the following theorem is based on some ideas of the paper:
K.I. Beidar and A.V. Mikhalev, Generalized polynomial identities and rings which are sums of two subrings, Algebra and Logic 34(1995), 1-5 (English translation).

Puczyłowski-Kępczyk (2001)

- Suppose $\mathfrak{F}$ is a homomorphically closed class of rings which is closed under direct powers. If every nonzero prime ring in $\mathfrak{F}$ contains a nonzero one-sided *PI* ideal, then $\mathfrak{F}$ consists of *PI* rings.

Let $R = R_1 + R_2$.

Puczyłowski-Kępczyk (2001)

(i) If R_1 or R_2 is a one-sided ideal of R and both R_1 and R_2 are PI rings, then R is a PI ring.

(ii) If R_1 is nil of bounded index (i.e. it satisfies the identity $x^n = 0$) and R_2 is a PI ring, then R is a PI ring.

Assume that A is an associative K-algebra and B, C are its subalgebras such that

$$A = B + C$$

(we keep this notation throughout the talk).

We say that an algebra H almost satisfies certain property w if it has an ideal I of finite codimension in H (i.e. $\dim_K H/I < \infty$) which satisfies w.

Consider two arbitrary classes $\mathbb{S}$ and $\mathbb{T}$ of algebras, for which $0 \in \mathbb{S}$ and $0 \in \mathbb{T}$. The class of all algebras H containing an ideal I such that $I \in \mathbb{S}$ and $H/I \in \mathbb{T}$ will be denoted by $\mathbb{ST}$, that is

$$\mathbb{ST} = \{H \,|\, \text{there exists } I \lhd H \text{ such that } I \in \mathbb{S} \text{ and } H/I \in \mathbb{T}\}.$$

Obviously $\mathbb{S} \subseteq \mathbb{ST}$ and $\mathbb{T} \subseteq \mathbb{ST}$.
We denote the class of all finite dimensional algebras, nilpotent algebras, nil of bounded index algebras and commutative algebras by $\mathcal{F}$, $\mathcal{N}$, $\mathcal{B}$ and $\mathcal{C}$, respectively.

$\mathcal{F}$ - the class of all finite dimensional algebras
$\mathcal{N}$ - the class of all nilpotent algebras
$\mathcal{B}$ - the class of all nil of bounded index algebras
$\mathcal{C}$ - the class of all commutative algebras

Let $A = B + C$.

Kępczyk (2008)

(i) If $B \in \mathcal{NF}$ and $C \in \mathcal{NF}$, then $A \in \mathcal{NF}$.

(ii) If $B \in \mathcal{BF}$ and $C \in \mathcal{BF}$, then $A \in \mathcal{BF}$.

(iii) If $B \in \mathcal{CF}$ and $C \in \mathcal{CF}$, then $A \in \mathcal{NCF}$.

Let f and g be polynomials such that if $R = R_1 + R_2$ and R_1 satisfies the identity $f = 0$ and R_2 satisfies the identity $g = 0$ then R is a PI ring. Moreover let $\mathcal{S}$ and $\mathcal{T}$ be the classes of all K-algebras satisfying identities $f = 0$ and $g = 0$, respectively.

Let $A = B + C$.

Kępczyk (2015)

- If $B \in \mathcal{SF}$ and $C \in \mathcal{TF}$, then A is a PI algebra.

Corollary:

- If $B \in \mathcal{NCF}$ and $C \in \mathcal{NCF}$, then A is a PI algebra.

Kępczyk (2016)

- Let $\mathcal{M}$ be the class of all semisimple finite dimensional K-algebras of the form $A = B + C$, where B satisfies the identity $h = 0$ and C satisfies the identity $w = 0$. Then all algebras in $\mathcal{M}$ satisfy a common polynomial identity.

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