Directed Jónsson and Gumm terms

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Jónsson terms

- B. Jónsson: terms for CD varieties

$$J_1(x,x,y) \approx x, \qquad J_{2k+1}(x,y,y) \approx y,$$
$$J_i(x,y,x) \approx x, \qquad 1 \le i \le n,$$
$$J_i(x,y,y) \approx J_{i+1}(x,y,y) \qquad 1 \le i \le 2k+1 \text{ odd},$$
$$J_i(x,x,y) \approx J_{i+1}(x,x,y) \qquad 1 < i < 2k+1 \text{ even}.$$

- Call this J(k).
- Kozik motivated by Barto: How about this directed variant?

$$d_1(x,x,y) \approx x, \qquad d_n(x,y,y) \approx y$$
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- Directed terms are simpler to describe,. . .
- . . . follow from near-unanimity in a more straightforward way, . . .
- . . . allow us to get undirected (classical) terms easily. . .
- . . . and make some proofs more comfortable:
- L. Barto's proof of Valeriote conjecture: Finite $\mathbf{A}$ generates a CM variety and is finitely related $\Rightarrow$ $\mathbf{A}$ has few subpowers.
- L. Barto, AK: Let $\mathbf{A}$ be a finite idempotent algebra. Then $B \trianglelefteq \mathbf{A}$ iff $B \trianglelefteq_{\mathrm{J}} \mathbf{A}$ (directed Jónsson absorption) and there is no B-blocker in $\mathbf{A}$.

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- A $(k+1)$-*box* from c to b and d, denoted by $B(c; b, d)$, is this:

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Suppose that $B(c; b, d)$ is a $(k+1)$-box. Then $c \to d$.

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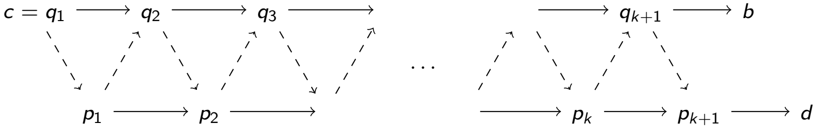

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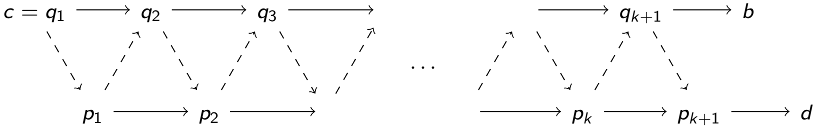

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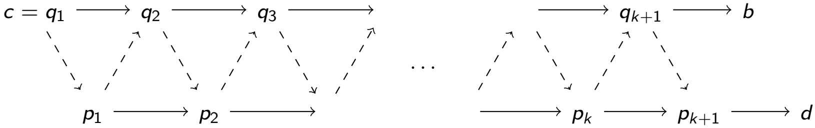

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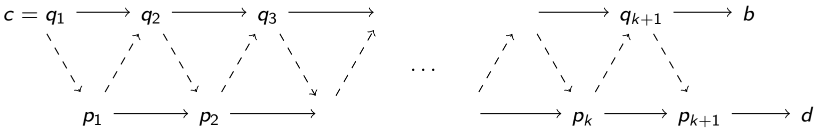

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- An *m-fence* from c to d is a sequence

- We have seen that there is a k-fence from x to z, but we can do better!

Lemma

For each $0 \leq i < k$ there exists a $(k-i)$-fence from x to z.

- Once we prove this lemma, we get a 1-fence from x to z which is almost as good as $x \longrightarrow z$ (see next two slides).

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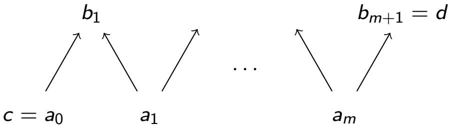

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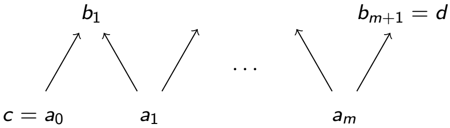

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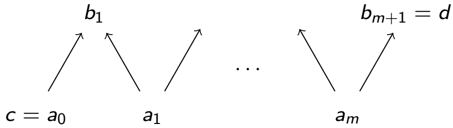

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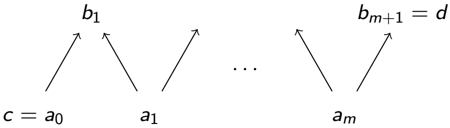

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Assume that there is a 1-fence $x \to b \leftarrow a \to d$. Then for every $\ell > 1$ there is an ℓ-box $B(x; b, d(b, d))$.

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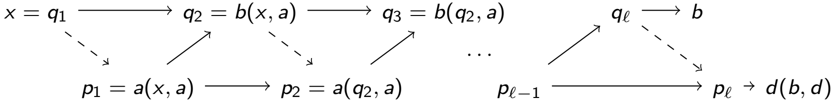

1-fence is enough

- The general pattern: Go from 1-fence to $(k+1)$-box to "$\rightarrow$."

Lemma

For each $0 \leq i < k$ there exists a $(k-i)$-fence from x to z.

- We now see that if we have a 1-fence $x \longrightarrow b \longleftarrow a \longrightarrow z$, we can get a $(k+1)$-box $B(x; b, z(b,z)) = B(x; b, z)$.

Lemma

Suppose that $B(c; b, d)$ is a $(k+1)$-box. Then $c \rightarrow d$.

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Lemma

For each $0 \leq i < k$ there exists a $(k-i)$-fence from x to z.

- Proof by induction on i. For 0, the Jónsson chain witnesses the fence.
- Assume we have

$$x \to b_1 \leftarrow a_1 \to b_2 \leftarrow a_2 \to \cdots \leftarrow a_{k-i} \to b_{k-i+1} \leftarrow a_{k-i+1} \to z.$$

- Applying the 1-fence to box to "$\to$" on the initial segment

$$x \to b_1 \leftarrow a_1 \to b_2,$$

  we get that $x \longrightarrow b_2(b_1, b_2)$.
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Intermezzo: Left squares

Lemma

If $c \to d$ and $e \dashrightarrow f$, then $c(e,f) \to d(e,f)$.

- It is enough to show this for $(c,d) \in E$.
- Since $(c,d) \in E$, there is a term s such that $s(\longrightarrow, \dashrightarrow, \longrightarrow) \subset \longrightarrow$ and
- For $c(x,z) \in \mathbf{F}(x,z)$, we define $c^2 = c(x, c(x,z))$.
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- We have:

$$x \to b_1 \leftarrow a_1 \to b_2 \leftarrow a_2 \to \cdots \leftarrow a_{k-i} \to b_{k-i+1} \leftarrow a_{k-i+1} \to z.$$

- Also, $x \longrightarrow b_2(b_1, b_2)$ and $b_2(b_1, b_2) \leftarrow b_2(x, b_2)$.
- We now claim that this is a shorter fence (note $z^2 = z$):

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- Here's how we build the begining of the fence:

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Summing it up

- We have shown, using fences and boxes, that $\forall k\, \exists n$, $\mathrm{J}(k) \Rightarrow \mathrm{DJ}(n)$.
- Our proof is constructive, but the n we get is big – roughly k^k.
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- The method works for Gumm terms (or Jónsson absorption), too.
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