Representing groups by endomorphisms of the random graph

Igor Dolinka

dockie@dmi.uns.ac.rs

Department of Mathematics and Informatics, University of Novi Sad

92nd Workshop on General Algebra (AAA92)

Prague, Czech Republic, 28 May 2016

This talk is dedicated to...

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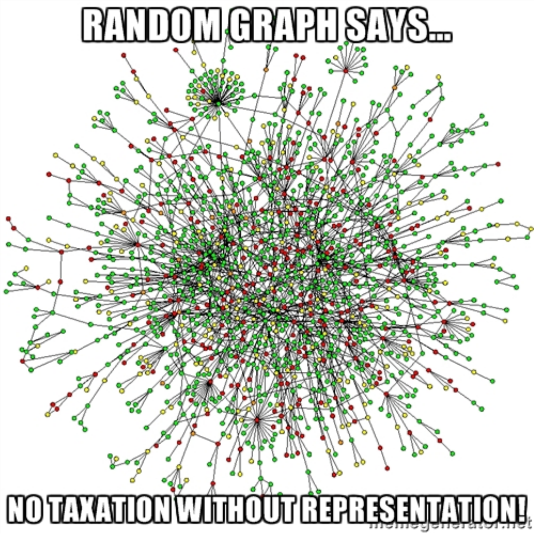
...my very first encounter with alcohol – and beer in particular – almost exactly 30 years ago (on the evening of 30 April 1986, to be exact) in a certain pub/brewery in Prague.

Representation is an important issue




SO, WHO IS THIS RANDOM GRAPH CHARACTER?
I HATE IT ALREADY
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Ready for take-off: Homogeneous structures

Let $\mathcal{A}$ be a (countable) first order structure. $\mathcal{A}$ is said to be (ultra)homogeneous if any isomorphism

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Remark

If we restrict to relational structures, 'finitely generated' becomes simply 'finite'.

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A class of finite(ly generated) structures with such properties is called a Fraïssé class.

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Theorem (Fraïssé)

Let $\mathbf{C}$ *be a Fraïssé class. Then there exists a unique countably infinite ultrahomogeneous structure* $\mathcal{F}$ *such that* $\mathrm{Age}(\mathcal{F}) = \mathbf{C}$.

Fraïssé theory (continued)

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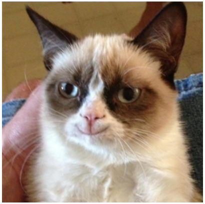


ENDOMORPHISM MONOIDS?
MMM, DELICIOUS...
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- overt: as maximal subgroups of S;
- covert: as Schützenberger groups (of $\mathscr{D}$-classes of S)

Sometimes, there is a very fine line between overt and covert... ☺

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IGOR!
Y U NO SEMIGROUP THEORY?
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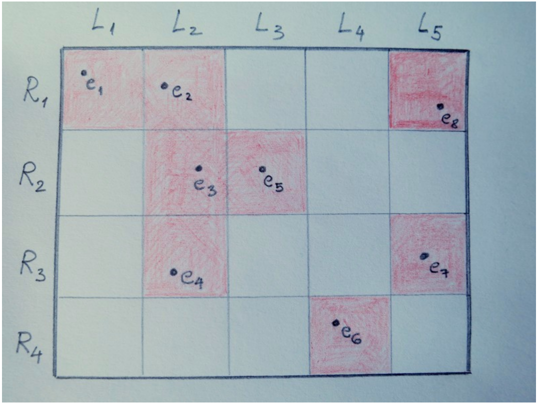
The eggbox picture of a $\mathscr{D}$-class



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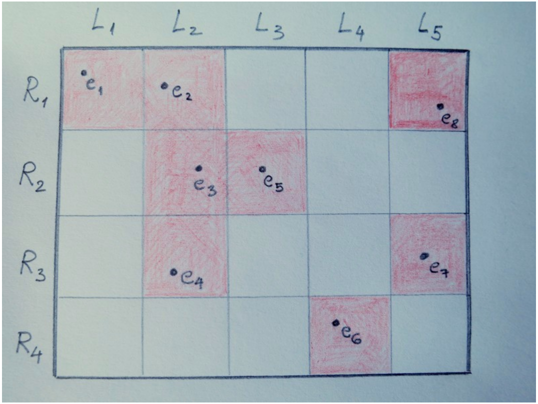

Groups (overt): $\mathscr{H}$-classes shaded red (these are all isomorphic)

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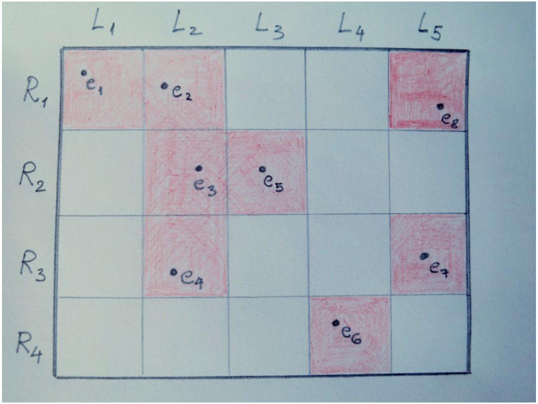

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maximal subgroups of a semigroup = $\mathscr{H}$-classes containing idempotents

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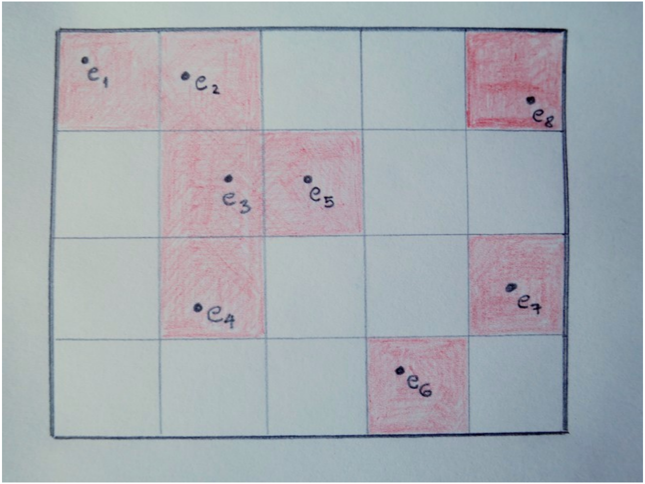
Hence, a is regular $\iff$ $a \mathrel{\mathscr{D}} e$ for and idempotent e.

A regular $\mathscr{D}$-class



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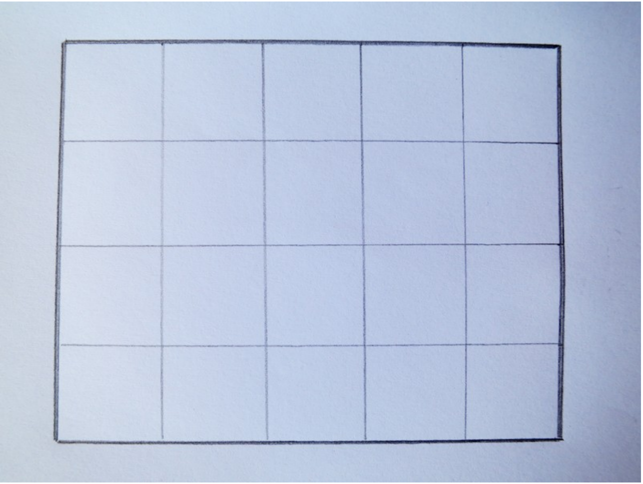
A non-regular $\mathscr{D}$-class



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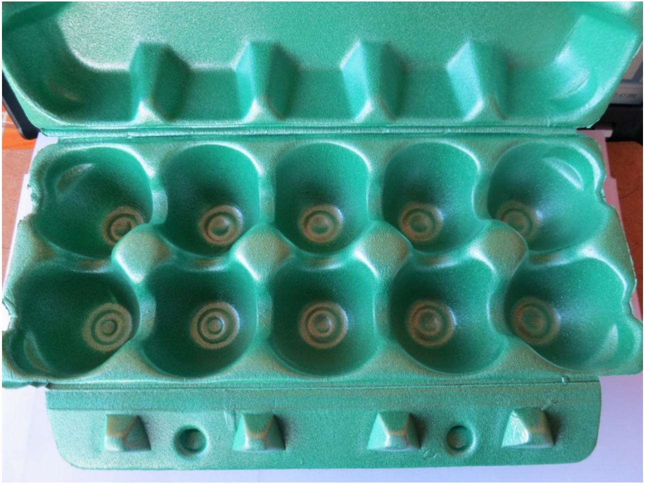

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If H is a group (so that D is regular), then $S_H \cong H$.

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(5) if $e = e^2$ and $\mathrm{rank}(e) = k$, then $H_e \cong \mathbb{S}_k$;

A classical example: $\mathcal{T}_X$

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(6) $\mathcal{T}_X$ is regular.


COULD WE GO
BACK TO ENDOMORPHISM MONOIDS, PLEASE?

End($\mathcal{A}$)

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We must be careful with the notion of an 'image' of an endomorphism if our language contains relational symbols, because besides $\mathrm{im}(f)$ we also have $\langle Af \rangle$, the induced substructure of A on Af.

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Lemma

$f \mathrel{\mathscr{D}} g \implies \langle Af \rangle \cong \langle Ag \rangle$.

Regular elements in End($\mathcal{A}$)

Proposition (Magill, Subbiah, 1974)

If $f \in End(\mathcal{A})$ is regular, then $\mathrm{im}(f) = \langle Af \rangle$.

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Schützenberger groups in End($\mathcal{A}$)

Proposition

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Schützenberger groups in $\mathrm{End}(\mathcal{A})$

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(i) *If* $t \in T_H$*, then* $t|_{Af}$ *is an automorphism of both* $\langle Af \rangle$ *and* $\mathrm{im}(f)$*;*

(ii) *the mapping* $\phi : \rho_t \mapsto t|_{Af}$ *is an embedding of* S_H *into* $\mathrm{Aut}(\langle Af \rangle) \cap \mathrm{Aut}(\mathrm{im}(f))$*.*

So, what the heck are the images of (idempotent) endomorphisms of Fraïssé limits?

Call a Fraïssé class **C** neat if it consists of finite structures, and for each $n \geq 1$ the number of isomorphism types of n-generated structures in **C** is finite.

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Examples:

- relational structures
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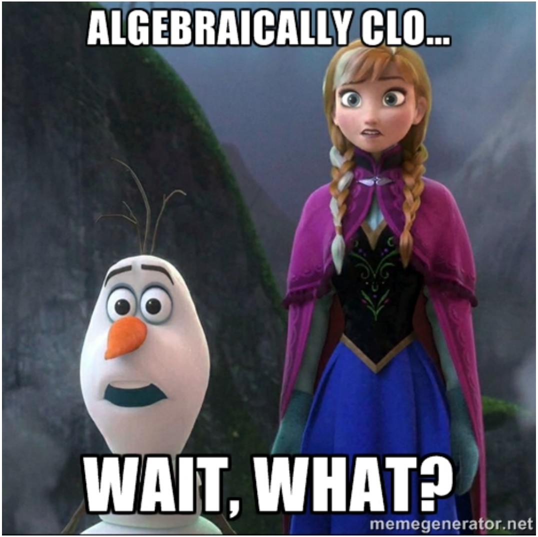
Theorem (ID, 2012)

Let **C** *be a neat Fraïssé class enjoying the strict AP and the 1PHEP. Then there exists and (idempotent) endomorphism f of F, the Fraïssé limit of* **C**, *such that $\mathcal{A} \cong \mathrm{im}(f)$ if and only if $\mathcal{A}$ is algebraically closed in $\overline{\mathbf{C}}$.*


ALGEBRAICALLY CLO...
WAIT, WHAT?
memegenerator.net

Algebraically closed stuff

Algebraically closed stuff

An L-formula $\Phi(\mathbf{x})$ is primitive if it is of the form

$$(\exists \mathbf{y}) \bigwedge_{i<k} \Psi_i(\mathbf{x}, \mathbf{y})$$

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Let $\mathbf{K}$ be a class of L-structures. An L-structure $\mathcal{A}$ is existentially (algebraically) closed (in $\mathbf{K}$) if for any primitive (positive) formula $\Phi(\mathbf{x})$ and any tuple $\mathbf{a}$ from A we have already $\mathcal{A} \models \Phi(\mathbf{a})$ whenever there is an extension $\mathcal{A}' \in \mathbf{K}$ of $\mathcal{A}$ such that $\mathcal{A}' \models \Phi(\mathbf{a})$.

Graphs

Countable e.c. graphs: R (Alice's Restaurant property)

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- the random digraph,
- the random bipartite graph,
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Proposition

A countable graph (V, E) is a.c. if and only if there exists $E' \subseteq E$ such that $(V, E') \cong R$ (that is, it is e.c.).

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Proposition

A countable graph (V, E) is a.c. if and only if there exists $E' \subseteq E$ such that $(V, E') \cong R$ (that is, it is e.c.). Consequently, for any a.c. graph Γ there is a bijective homomorphism $R \to \Gamma$.


WOW, COOL!
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Frucht's Theorem (1939)

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Name of the game: Strengthen this for countable a.c. graphs.

The team

Point Guard:
Martyn Quick

Center:
Jillian “Jay” McPhee

Shooting Guard:
Robert “Bob” Gray

Forward:
“Baby” James Mitchell

Power Forward:
Dr. D

Automorphism groups of countable a.c. graphs

Theorem

Let Γ be a countable graph. Then there exist $2^{\aleph_0}$ pairwise non-isomorphic countable a.c. graphs whose automorphism group is $\cong \mathrm{Aut}(\Gamma)$.

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Proof. For a (simple) graph Δ, let $\Delta^\dagger$ denote its complement.

- $\mathrm{Aut}(\Delta^\dagger) = \mathrm{Aut}(\Delta)$.

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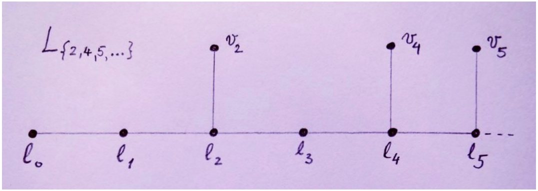
- Δ any graph, Λ infinite locally finite graph $\Rightarrow (\Delta \uplus \Lambda)^\dagger$ is a.c.
- The central idea – consider l.f. graphs L_S for $S \subseteq \mathbb{N} \setminus \{0, 1\}$:



Automorphism groups of countable a.c. graphs

Proof (cont'd).

- Properties of L_S ($S, T \subseteq \mathbb{N} \setminus \{0,1\}$):
  - Each L_S is rigid ($\mathrm{Aut}(L_S) = \mathbf{1}$).
  - $L_S \cong L_T \iff S = T$.

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- If L_S is isomorphic to no connected component of Γ (and this excludes only countably many choices of S), then

$$\mathrm{Aut}(\Gamma \uplus L_S)^{\dagger} = \mathrm{Aut}(\Gamma \uplus L_S) \cong \mathrm{Aut}(\Gamma) \times \mathrm{Aut}(L_S) \cong \mathrm{Aut}(\Gamma).$$

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- $S_1 \neq S_2$ yield non-isomorphic a.c. graphs.

Images of idempotent endomorphisms

Theorem (Bonato, Delić, 2000; ID, 2012)

Let Γ *be a countable graph. There exists an idempotent* $f \in \mathrm{End}(R)$ *such that* $\mathrm{im}(f) \cong \Gamma$ *if and only if* Γ *is a.c.*

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Theorem

If Γ is a countable a.c. graph, then there exists an (induced) subgraph $\Gamma' \cong \Gamma$ of R such that there are $2^{\aleph_0}$ idempotent endomorphisms f of R such that $\mathrm{im}(f) = \Gamma'$.

The number of regular $\mathscr{D}$-classes with a given group $\mathscr{H}$-class

Theorem

(i) *Let* Γ *be a countable graph. Then there exist* $2^{\aleph_0}$ *distinct regular* $\mathscr{D}$*-classes of* $\mathrm{End}(R)$ *whose group* $\mathscr{H}$*-classes are* $\cong \mathrm{Aut}(\Gamma)$.

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Corollary

$\mathrm{End}(R)$ *has $2^{\aleph_0}$ regular $\mathscr{D}$-classes.*

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Corollary

$\mathrm{End}(R)$ *has* $2^{\aleph_0}$ *regular* $\mathscr{D}$*-classes. (You know, the ones with eggs...)*

The size of a regular eggbox

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Every regular $\mathscr{D}$-class of $\mathrm{End}(R)$ *contains* $2^{\aleph_0}$ *many* $\mathscr{R}$*- and* $\mathscr{L}$*-classes.*

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However, all these idempotents are not $\mathscr{R}$-related.

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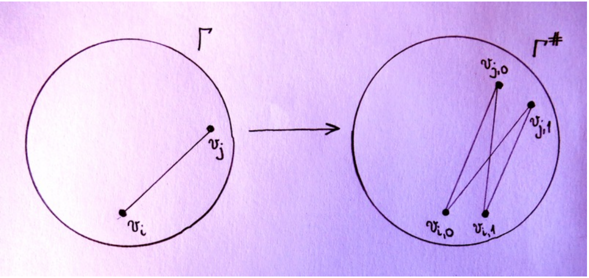

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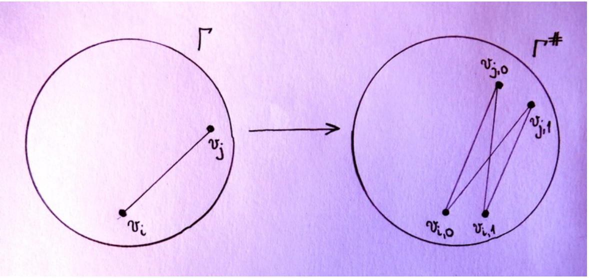

Construct R around $\Gamma^\sharp$, so that $R = R_{\Gamma^\sharp}$.

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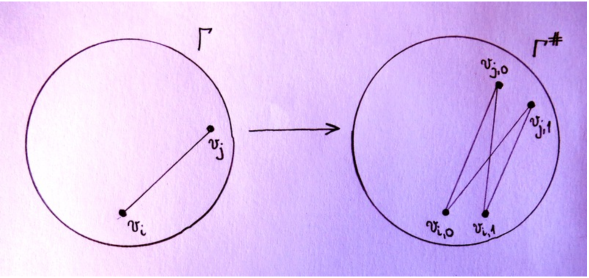

Construct R around $\Gamma^\sharp$, so that $R = R_{\Gamma^\sharp}$.

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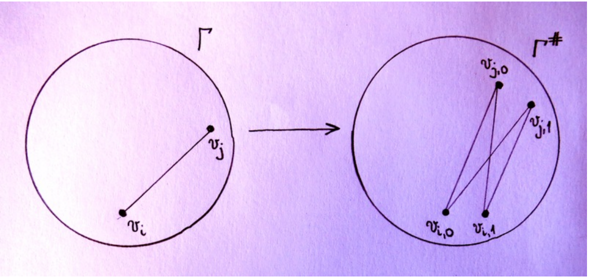

Construct R around $\Gamma^\sharp$, so that $R = R_{\Gamma^\sharp}$.

Γ a.c. $\Longrightarrow$ $\Gamma^\sharp$ a.c. Hence, the identity map on $\Gamma^\sharp$ can be extended to an endomorphism $g : R \to \Gamma^\sharp$.

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For each binary sequence $\mathbf{b} = (b_i)_{i \in \mathbb{N}}$ define a map $\psi_{\mathbf{b}}$ on $\Gamma^\sharp$ by

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$g\psi_{\mathbf{b}} \in \mathrm{End}(R)$ are idempotents, $\mathrm{im}(g\psi_{\mathbf{b}}) \cong \Gamma \Rightarrow$ all these idempotents are $\mathscr{D}$-related to e.

Different images $\Rightarrow$ they are not $\mathscr{L}$-related.

Non-regular eggboxes

Theorem

Let $\Gamma \ncong R$ be a countable a.c. graph. Then there exists a non-regular endomorphism of R such that $\mathrm{im}(f) \cong \Gamma$ and D_f contains $2^{\aleph_0}$ many $\mathscr{R}$- and $\mathscr{L}$-classes.

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Open Problem
Are there any non-regular eggboxes of some other size?

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Proposition

Let f be an injective endomorphism of $R = (V, E)$ as described above, with $Vf = V_0$. Then

$$S_{H_f} \cong \mathrm{Aut}(\langle V_0 \rangle) \cap \mathrm{Aut}(\mathrm{im}(f))$$

Schützenberger groups in End(R)

So, to show a universality result for Schützenberger groups in End(R), one needs to extend the Frucht-de Groot-Sabidussi Theorem to countable a.c. graphs with 2-coloured edges (blue and red, say) where the 'red graph' is $\cong R$.

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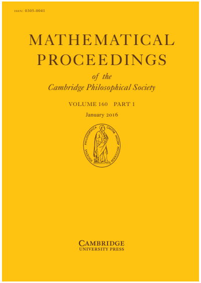
Theorem

Let Γ be any countable graph. There are $2^{\aleph_0}$ non-regular $\mathscr{D}$-classes of End(R) *such that the Schützenberger groups of the $\mathscr{H}$-classes within them are* $\cong \mathrm{Aut}(\Gamma)$.

Reference



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THANK YOU!

Questions and comments to:

dockie@dmi.uns.ac.rs

Further information may be found at:

http://people.dmi.uns.ac.rs/∼dockie