Finer complexity of Constraint Satisfaction Problems with Maltsev Templates

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May 24, 2016

Constraint Satisfaction Problems

Definition

The constraint satisfaction problem (CSP) is the combinatorial decision problem with

Instance: a quadruple $(V; \mathcal{A} = \{A_i \ : \ i \in I\}; \sigma; \mathcal{C})$, where

- V - set of variables;
- $\mathcal{A}$ is a collection of domains (values) for variables in V;
- $\sigma : V \to I$ is a sort function;
- $\mathcal{C}$ is a set of constraints, $\{C_1, \ldots, C_q\}$, where each constraints associates an m-ary relation

$$R_i \subseteq A_{\sigma(v_1)} \times \cdots \times A_{\sigma(v_m)}$$

to an m-tuple of variables $(v_1, \ldots, v_m) \in V^m$.

Definition

(continued...)

Question: Does there exists a solution, i.e. a function

$$f : V \to \bigcup_{i \in I} A_i$$

so that each variable is mapped to its domain and the tuples of assigned values satisfy all constraints R?

- Generally, we may assume that all $A_i = A$, for some fixed set A, and recast the CSP as a homomorphism problem for a finite relational structure $\mathbb{A}$ in a finite relational vocabulary.
- Clearly, the general problem is in NP.

Dichotomy Conjecture (T. Feder, M. Vardi, 1998): For any finite relational template $\mathbb{A}$, $CSP(\mathbb{A})$ is either in P or it is NP-complete.

Generalized Gaussian Elimination

- Maltsev templates: templates admitting a ternary polymorphism that satisfies
$$m(xxy) = m(yxx) = y.$$
- (Dalmau-Bulatov, 2006): every CSP with a Maltsev template is in P, since it can be solved by Generalized Gaussian Elimination
- P. Idziak, P. Markovic, R. McKenzie, M. Valeriote, R. Willard (2010) generalized this to the templates with k-cube terms

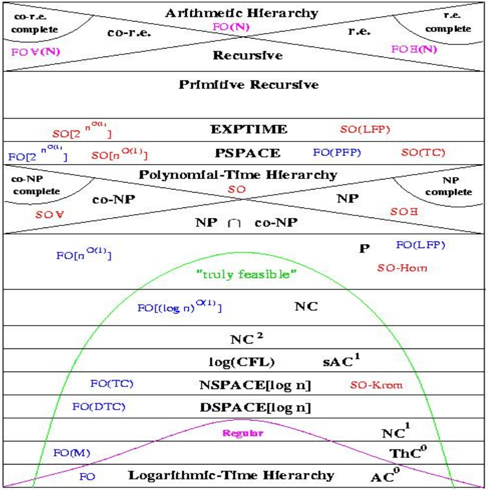

Figure: Descriptive Complexity Hierarchy

- Bulatov-Dalmau algorithm is based on the idea of processing constraints one at a time and “shrinking" the solution subuniverse.
- The efficiency of the algorithm hinges on the fact that the solution subuniverse can be compactly represented, i.e. that it has a generating set of size polynomial in n.

Definition

Suppose $B \subseteq A^n$

$$\mathrm{Fork}(B) = \{(i, b, c) \in [n] \times A \times A \,|\, \exists \mathbf{u}, \mathbf{v} \in B \text{ with } u_j = v_j \text{ for all } 1 \leq j < i \text{ and } u_j = b, v_j = c\}.$$

A subset $T \subseteq B$ is called a compact representation of B if Fork $(T) =$ Fork (B) and T is minimal with respect to this property.

Theorem

(A. Habte, 2015 [MSc thesis]) Solvability of CSPs with Maltsev templates is expressible in FP+RK*.*

Proof is a straightforward but tedious encoding of the Bulatov-Dalmau algorithm.

Revisiting the Bulatov-Dalmau algorithm, what is really needed are the forking pairs where (c, d) generate strictly simple (i.e. minimal) subalgebras of A. The downside is that the generation property is lost but, we do not need to keep track of *all* solutions anyway, if we are only interested in the decision problem.

Atomization

- Suppose $\mathbb{A}$ is a finite template admitting a Maltsev polymorphism m.
- Let $\mathbb{X} = \{x_1, x_2, \ldots, x_n\}$ be the input structure.
- Even though, originally, the domain for each x_i is A, we will work within a more general framework: the domain for x_i is A_i, which is a subuniverse of $\mathbb{A}$

We want to prove the following:

Theorem

The solvability of a 1-consistent multisorted Maltsev CSP with domains $A_1, \ldots, A_n$ is equivalent to a disjunction of solvability problems over polynomially many (in n) 1-consistent multisorted CSPs whose domains are strictly simple Maltsev subuniverses $B_i \leq A_i$.

- The proof will be by induction on

$$M = \max(|A_i| \, : \, i \in [n]\}$$

- What we assume is that on smaller templates, which are jointly Maltsev, if the problem is solvable, we can detect that by examining at most $f(n)$ problems over strictly simple domains, for some polynomial f.
- Factor out the maximal ($\neq 1_{A_i}$) congruences θ_i in all A_i. The quotient structure

$$A_1/\theta_1 \times \cdots \times A_n/\theta_n$$

is still 1-consistent.

- Solve the problem in the quotient structure; if no solution, the original CSP cannot have one either. If there is a solution, decompose the problem into a disjunction of polynomially many (in n) parallel smaller problems and apply the inductive hypothesis

Simple Domains Case

- Suppose $A_1 \times \cdots \times A_n$ is a subdirect product of simple Maltsev algebras, which is 1-consistent
- For every pair of indices i, j, the restriction to $A_i \times A_j$, is one of the following two types:
  1. $A_i \times A_j$
  2. an isomorphism $f : A_i \to A_j$
- We can define an equivalence relation on $[n]$, and consistently choose minimal sets B_i (in the sense of TCT) for each A_i.
- B_i's are Maltsev subalgebras in their own right.
- The solvability of the problem over A_i's is equivalent to the solvability of the problem over B_i's.

Strictly Simple Maltsev Case

Two possibilities:

1. Simple affine algebras
2. Quasiprimal algebras

The invariant relations in the products of algebras of type (1) can be characterized using systems of linear equations over finite fields.

The CSP restricted to the product of algebras of type (2) can be FO reduced to the CSP over a 2-element Boolean algebra which is FO-definable and consisting of conjunctions of equations of the form $x_i = x_j$.

Descriptive Complexity Consequence

- The reduction to the strictly simple cases does not require a logic stronger than first-order (all subuniverses and congruence relations are pp-definable with parameters)
- The problem reduces to solving a disjunction of polynomially many conjunctions of systems of linear equations over finite fields $\mathbb{Z}_p$.

Theorem

(D., Habte, 2016) 1-consistent CSPs over finite Maltsev templates are definable in $\mathrm{FO+R}\kappa_{p_1,\ldots,p_r}$*, where* $p_1, \ldots, p_r$ *is a set of finitely many distinct primes, dependent on* $\mathbb{A}$ *only.*

Definition

(Arithmetical Nondeterministic Logspace Classes) Let $k \in \mathbb{N}$. A language $L \subseteq \Sigma^*$ belongs to $\mathrm{MOD}_k L$ if there is a nondeterministic logspace machine M, such that $x \in L$ if and only if the number of non-accepting computations for x is not divisible by k.

Theorem

(Holm, 2010) The logic $\mathrm{FO{+}R\kappa}_p$ *captures the complexity class* $\mathrm{MOD}_p L$ *on finite ordered structures.*

Corollary

(D., Habte, 2016) 1-consistent CSPs over finite Maltsev templates are in the complexity classes $\mathrm{MOD}_k L$*, where* $k = p_1 \ldots p_r$*, for some distinct primes* $p_1, \ldots, p_r$*.*

An earlier result of Larose and Tesson states that If $V(\mathbb{A})$ omits the unary type but admits the affine type, then there exists a prime p such that $CSP(\mathbb{A})$ is $\mathrm{MOD}_p L$-hard under FO reductions.