# Application of Lattice-theoretical Methods to the existence of equilibrium in ordinal games

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**Abstract**

In this work we prove the existence of Berge's strong equilibrium in non cooperative and quasi-supermodular ordinal games. Also, we show that the set of Berge's strong equilibrium is a complete lattice.
**Keywords:**Berge's strong equilibrium, Fixed point theorems, Lattice, Nash equilibrium, Ordinal games, Quasi-supermodular games.

## 1 Introduction

The Berge's strong equilibrium ([2]) is one of Nash's equilibrium ([9]) refinements which is stable with respect to the deviation of all players except one. Earlier contributions on the existence of Berge's strong equilibrium were based on the Kakutani or Browder-Fan fixed point theorems ([1], [4] and [8]) of correspondences. This requires a topological structure on the strategy space of players, convexity and compactness of the strategy subset of each player. Also, the payoff function of each player is quasi-concave and continuous. Recently, ([10]) proves the existence of Berge's strong equilibrium and studies its order structure for non-cooperative quasi-supermodular games (the strategy sub-set of each player is a lattice and its preferences are represented by a quasi-modular payoff function). The application of Zhou's fixed point theorem ([11]) instead of classical fixed point avoids the use of standard assumptions on the game The purpose of this work is to improve the results in ([10]) to the ordinal games (the strategy sub-set of each player is a lattice and its preferences are represented by a quasi-modular weak order on his strategy sub-set). We introduce the notion of quasi-supermodular ordinal games to the context of Berge's strong equilibrium and we prove the existence of Berge's strong equilibrium which turns out to be a complete lattice. This paper is organized as follows: In Section 2, we presente definitions and theorem of Zhou ([11]) that are used throughout this work. In Section 3, we define the Berge's strong equilibrium in ordinal games. In section 4, we study the existence of the Berge's strong equilibrium in a quasi-supermodular ordinal games. Finally, in section 5 we give summary of our study.

# 2 Lattice sets

**Definition 1** *An ordered set $X$ is said to be a lattice if it contains the supremum and the infinimum of each pair of its elements.*

**Definition 2** *A lattice set $X$ is said to be complete if each nonempty sub-set of $X$ has a supremum and the infinimum. A sub-set $A$ of a lattice $X$ is called a sub-complete sub-lattice of $X$ if each nonempty sub-set $B$ of $A$ admits a supremum and infinimum in $A$.*

**Definition 3** *Let $X$ a lattice and $F : X \longrightarrow X$ a correspondence or multi-valued function. $F$ is said to be a Veinott-increasing if for each $x, y \in X$ such that $x < y, a \in F(x)$ and $b \in F(y)$ then $a \wedge b \in F(x)$ and $a \vee b \in F(y)$.Finally, the set of fixed points of the correspondence $F : X \longrightarrow X$ is defined as $\{x \in X : x \in F(x)\}$.*

Next, we give a fixed point theorem of Zhou in lattice sets which plays an important role for the existence of the Berge's strong equilibrium in supermodular games. Also, we recall the definition of the interval topology on the lattice set.

**Theorem 1** *Zhou ([11]) Let $X$ a nonempty complete lattice, and $F : X \rightarrow X$ a correspondence with nonempty valued. If $F$ is Veinott increasing and for each $x$ in $X$, $F(x)$ is a subcomplete lattice of $X$, then the fixed point set of $F$ is nonempty complete lattice.*

**Definition 4** *Let $(X, \succeq)$ a lattice and $a, b \in X$. We call closed intervals in $X$, the subsets of the form $[a, b] = \{x \in X : b \succeq x \succeq a\}$, , $[a, \infty) = \{x \in X : x \succeq a\}$, or $(-\infty, b] = \{x \in X : b \succeq x\}$.*

**Definition 5** *We say that a lattice $(X, \succeq)$ is endowed with the inerval topology if each closed subset of $(X, \succeq)$ can be represented as the intersection of sets that are finite unions of closed intervals in $(X, \succeq)$.*

# 3 Berge's strong equilibrium in ordinal games

Let us consider the following ordinal noncooperative game in normal form:

$$G = \left(I, \ \ (X_i, \ \ \succeq_i)_{i \in I}\right)$$

where $I = \{1, ..., n\}$ is a finite set of players, $X_i$ is the set of strategies of player $i$ and, if $X = \prod_{i=1}^{n} X_i$ denotes the set of issues(joint strategy) of the game $G$, $\succeq_i$is a binary relation on $X = \prod_{i=1}^{n} X_i$ of $i$ which reflects his preferences over the outcomes of the game $G$. Each of binary relation $\succeq_i$is assumed to be a weak order(transitive and complete). For each player$i$, as usual, we let $I \backslash \{i\} = \{1, ..., i-1, i+1, ..., n\} = \{j \in I, j \neq i\}$ and we denote $X_{-i} = \prod_{j \neq i} X_j = \prod_{j \in I \backslash \{i\}} X_j$ and if $x \in X = \prod_{i=1}^{n} X_i, x_{-i} = (x_j)_{j \neq i} \in X_{-i}$. Choosing a strategy $x_i \in X_i$, the aim of each player in the game $G$ is to maximize his preferences over the set $X = \prod_{i=1}^{n} X_i$. Recall that $x \in X$ is a Nash equilibrium of the game $G$ if for every $i \in I$, for all $x_i \in X_i, \nexists \ z_i \in X_i$ such that $(z_i, x_{-i}) \succ_i (x_i, x_{-i})$ where $\succ_i$is the asymmetric part of $\succeq_i$. In the following definition, we extend the definition of the cardinal Berge's strong equilibrium ( [2]) to the ordinal games.

**Definition 6** *Berge ( [2])* ***A Berge's strong equilibrium*** *of the game G is an n-tuple of strategies* $\bar{x} \in X$ *such that* $\forall i \in I, \forall j \in I \backslash \{i\}, \nexists \ y_{-i} \in X_{-i}, (\bar{x}_{i,} y_{-i}) \succ_j (\bar{x}_{i,}, \bar{x}_{-i})$.

In other words, while at a Nash equilibrium no one of the players has interest to modify alone his strategy at a Berge's strong equilibrium, for each player $i$, it is the complementary coalition which has no interest to deviate.

**Remark 1** *The set of Berge's strong equilibrium in ordinal games is included in the set of Nash equilibrium. To see this, let* $(\bar{x}_{i,}, \bar{x}_{-i})$ *a Berge's strong equilibrium. For each* $i \in I$, *take* $x_i \in X_{-i}$ *and* $j \in I \backslash \{i\}$ *.Since* $i \in I \backslash \{j\}$ *,then the relation* $(\bar{x}_{j,} \bar{x}_{I \backslash \{i,j\}}, x_i) \succ_i (\bar{x}_{i,}, \bar{x}_{-i})$ *is impossible. Because,* $(\bar{x}_{j,} \bar{x}_{I \backslash \{i,j\}}, x_i) = (x_{i,}, \bar{x}_{-i})$, *then* $(\bar{x}_{i,}, \bar{x}_{-i})$ *is a Nash equilibrium. It is worth nothicing that these two concepts coincide in games with two persons.*

Now, we define the best reply correspondence to the context of Berge's strong equilibrium ([7]). For each $i$, let us denote by $\succeq_{-i}$ the vector of the preferences of the members of $I \backslash \{i\}$ and call best reply correspondence for the complementary coalition $I \backslash \{i\}$, $\Gamma_{-i} : X \to X$ defined by:

$$\Gamma_{-i}(x) = \left\{y \in X : \nexists t_{-i} \in X_{-i} \quad (x_i, t_{-i}) \ \succ_{-i} \left(x_i, y_{-i}\right)\right\}$$

and for each $x \in X$, we set

$$\Gamma(x) = \bigcap_{i \in I} \Gamma_{-i}(x)$$

With these notations, a Berge's strong equilibrium is a fixed point of the correspondence $\Gamma$, that is an-tuple $\bar{x} \in \Gamma(\bar{x})$.

In the following subsections, we give analogous definitions of quasi-supermodular games and complement srategic property given in ordinal games for the Nash equilibrium to the context of ordinal Berge's strong equilibrium.

## 3.1 Quasi-supermodular order

**Definition 7** *Let $G = \left(I,\ (X_i,\ \succeq_i)_{i\in I}\right)$ a normal game where $(X_i, \geq_{i\in I})$ is a lattice. We say that $\succeq_i$is quasi-supermodular on $X_{-i}$ for each $x_i$. if for $t, z \in X$, we have the following conditions:*

$$(x_i, z_{-i}) \succeq_i (x_i, z_{-i} \wedge t_{_{-i}}) \Longrightarrow \left(x_i, z_{-i} \vee t_{_{-i}}\right) \succeq_i \left(x_i, t_{_{-i}}\right) \tag{1}$$

$$(x_i, z_{-i}) \succ_i (x_i, z_{-i} \wedge t_{_{-i}}) \Longrightarrow \left(x_i, z_{-i} \vee t_{_{-i}}\right) \succ_i \left(x_i, t_{_{-i}}\right) \tag{2}$$

## 3.2 Order with strategic complement property

**Definition 8** *We say that $\succeq_i$ satisfies the strategic complement property in $(x_i, x_{-i}) \in X_i \times X_{_{-i}}$ if for every $y,\ z, t \in X$ such that $z_{-i} \geq t_{-i}, y_i \geq x_i$ we have:*

$$(x_i, z_{-i}) \succeq_i (x_i, t_{-i}) \Longrightarrow (y_i, z_{-i}) \succeq_i (y_i, t_{-i})\,; \tag{3}$$

$$(x_i, z_{-i}) \succ_i (x_i, t_{-i}) \Longrightarrow (y_i, z_{-i}) \succ_i (y_i, t_{-i}) \tag{4}$$

## 3.3 Berge's strong equilibrium in quasi-supermodular ordinal games

**Definition 9** *An ordinal game $G = \left(I,\ (X_i,\ \succeq_i)_{i\in I}\right)$ is called quasi-supermodular if for each $i \in I, X_i$ is a lattice and $\succeq_i$verifies the equations (1)-(4).*

***Lemma 1*** *Let $G = \left(I,\ (X_i,\ \succeq_i)_{i\in I}\right)$ be a quasi-supermodular game. Then $\Gamma(x)$ is Veinott increasing in $x$.*

***Proof.*** *Let $x \leq y$ and $a \in \Gamma(x), b \in \Gamma(y)$, we prove that $a \wedge b \in \Gamma(x), a \vee b \in \Gamma(y)$. We have $a \wedge b \in X$ and $a \vee b \in X$ since $X_i$ is a lattice. By definition of $\Gamma(x)$ and $\Gamma(y)$, we get*

$$\forall i \in I, \forall j \in I \backslash \{i\}\,,\quad (x_i, a_{-i}) \succeq_j (x_i, a_{-i} \wedge b_{-i})$$

*and*

$$\forall i \in I, \forall j \in I\backslash\{i\}, \quad (y_i, b_{-i}) \succeq_j (y_i, a_{-i} \wedge b_{-i})$$

*Assume that* $a \leq\ b$, *then* $a \wedge b = a \in \Gamma(x)$ *and* $a \vee b = b \in \Gamma(y)$. *By supermodularity, we have*

$$\forall i \in I, \forall j \in I\backslash\{i\}, \quad (x_i, a_{-i} \vee b_{-i}) \succeq_j (x_i, b_{-i})$$

*and*

$$\forall i \in I, \forall j \in I\backslash\{i\}, \quad (y_i, a_{-i} \vee b_{-i}) \succeq_j (y_i, a_{-i})$$

*Since*

$$\forall i \in I, \forall j \in I\backslash\{i\}, \quad (x_i, a_{-i} \vee b_{-i}) \succeq_j (x_i, b_{-i})$$

*and* $x \leq\ y$, *it follow from the strategic complement property:*

$$(y_i, a_{-i} \vee b_{-i}) \succeq_i (y_i, b_{-i}) \ \ (y_i, a_{-i} \vee b_{-i}) \succeq_j (y_i, b_{-i})$$

*From* $b \in \Gamma(y)$, *we deduce that*

$$(y_i, a_{-i} \vee b_{-i}) \sim (y_i, b_{-i})$$

*Then* $a \vee b \in \Gamma(y)$. *Now, suppose that*

$$\forall i \in I, \forall j \in I\backslash\{i\}, \quad (x_i, a_{-i})_j \succ_j (x_i, a_{-i} \wedge b_{-i})$$

*By supermodularity, we have*

$$\forall i \in I, \forall j \in I\backslash\{i\}, \quad (x_i, a_{-i} \vee b_{-i}) \succ_j (x_i, b_{-i})$$

*Once again, by strategic complement property, it follows*

$$(y_i, a_{-i} \vee b_{-i}) \succ_j (y_i, b_{-i})$$

*We obtain a contradiction since* $(y_i, a_{-i} \vee b_{-i}) \sim (y_i, b_{-i})$. *Thus,*

$$(x_i, a_{-i})_j \sim (x_i, a_{-i} \wedge b_{-i})$$

*.Finally,* $a \wedge b\ \in\ \Gamma(x)$. *It remains to prove that for each* $x$, $\Gamma(x)$ *is a sublattice of* $X$*.Let* $a, b \in \Gamma(x)$. *By supermodularity and using the same arguments above, we obtain* $a \wedge b \in \Gamma(x)$ *and* $a \vee b \in \Gamma(x)$. ■

# 4 Existence of the Berge's strong equilibrium in a quasi-supermodular ordinal games

In this section, we give sufficient conditions for the existence of the Berge's strong equilibrium in quasi-supermodular ordinal games.

**Theorem 2** *Let the ordinal game* $G = \left(I,\ \ (X_i,\ \ \succeq_i)_{i\in I}\right)$ *be a quasi-supermodular. Assume the following assumptions on* $G = \left(I,\ \ (X_i,\ \ \succeq_i)_{i\in I}\right)$.

(1) $\forall i \in I$, $X_i$ *is nonempty and compact with respect of the interval topology;*

(2) $\forall i \in I, \forall j \in I \backslash \{i\}$, *the order* $\succeq_j$ *is upper semicontinuous on* $X_j$*;*

(3) $\forall x \in X, \Gamma(x) \neq \varnothing$.

*Then the set of the Berge's strong equilibrium of* $G = \left(I,\ \ (X_i,\ \ \succeq_i)_{i\in I}\right)$ *is nonempty complete lattice.*

***Proof.*** *The proof is inspired from ([5]) and ([6]) given for the Nash equilibrium. From Birkoff's theorem (See TheoremX.20, ([3]), we have a lattice is compact for the interval topology if and only if it is complete. It follows from hypothesis (1) that* $\forall i \in I$, $X_i$ *is complete. Also, we can prove that the product topology and the interval topology on* $X = \prod_{i=1}^{n} X_i$ *coincide. For more details, see ([5]) and ([6]). Once again, by Birkoff's theorem,* $X = \prod_{i=1}^{n} X_i$ *is a complete lattice since it is compact as a product of compact* $X_i$. *Now, we prove that the correspondence* $\Gamma(x) = \bigcap_{i\in I} \Gamma_{-i}(x)$ *where :*

$$\Gamma_{-i}(x) = \left\{ y \in X : \nexists t_{-i} \in X_{-i} \quad (x_i, t_{-i}) \ \succ_{-i} \left(x_i, y_{-i}\right) \right\}$$

*verifies the Zhou's fixed point. By Lemma 1,* $\Gamma(x)$ *is Veinott increasing in* $x$. *It remains to show that* $\Gamma(x)$ *is sub-complete sublattice of* $X$. *From the assumptions (1), (2) and Lemma 1 in ([6]), we have,* $\Gamma_{-i}$ *has nonempty and compact valued . It follows that* $\Gamma$*has compact values as intersection of* $\Gamma_{-i}$ *and nonempty values by assumption (3)..For* $x \in X, \Gamma(x)$ *is compact for the interval topology then. Let* $A \subseteq \Gamma(x)$, *then* $\sup A$ *belongs to* $X$*.From the definition of order topology, the sets* $[a, b] \cap \Gamma(x)$ *is a subbase of closed subsets for the toplogy of* $\Gamma(x)$. *Using the family* $[a, \sup A] \cap \Gamma(x)$ *and the finite property of intersection in the compact* $\Gamma(x)$ *,we claim that* $\sup A \in \Gamma(x)$. *The same is true for* $\inf A$*.Then,* $\Gamma(x)$ *is subcomplete sublattice of* $X$*.Finally, from Zhou's fixed ponit thoerm, the Berge's strong equilibrium of* $G = \left(I,\ \ (X_i,\ \ \succeq_i)_{i\in I}\right)$ *is nonempty complete lattice.* ■

# 5 Summary

In this work, we have extended the existence results of Berge's strong equilibrium in ([10]) to the ordinal quasi-supermodular games. It remains to weaken the compacity assumption on the stratery space of each player and catch some light on a potential applications of our results

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