### ON THE LATTICE STRUCTURE OF FOULIS SEMIGROUPS

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92. Arbeitstagung Allgemeine Algebra Prague, May 27–29, 2016

#### 1. FOULIS AND BAER SEMIGROUPS

# Definition [D.J.Foulis (1960)] A Baer \*-semigroup is an algebra (S, ., 0, \*, '), where (S, ., 0) is a semigroup with zero, \* is an involution (an idempotent semigroup antimorphism) every x' is a projection (self-adjoint idempotent), the principal ideal generated by x' is the right annihilator of the element x: x'y = y iff xy = 0.

The operation ' is necessarily unique.

The projections in its range are said to be *closed*;

the closed projections form an orthomodular lattice with orthocomplementation  $^\prime$  and ordering given by

 $p \le q$  iff pq = p (iff qp = p).

A dual definition.

The "right annihilator" axiom: x'y = y iff xy = 0.

The operation ' could equivalently be replaced by an operation ' where x' is a projection and satisfies the "left annihilator axiom"

• the principal ideal generated by x' is the left annihilator of x: yx' = y iff yx = 0.

The closed projections are the same: ran' = ran'.

- In 1960, D.Foulis proved the 'coordinatization theorem': every OM lattice is isomorphic to the lattice of closed projections of some Baer \*-semigroup.
- In 1972, T.S.Blyth and M.F.Janowitz suggested the term 'Foulis semigroup' for what Foulis called a Baer \*-semigroup.
- In 1973, D.H.Adams shew that Foulis semigroups form a vari-
- ety: the annihilator axiom is equivalent to the identity

x'y(xy)' = y(xy)'.

• In 1978, M.P.Drazin introduced, on a proper involution semigroup, the so called *star order*:

 $a \leq b$  iff  $a^*a = a^*b$  and  $aa^* = ba^*$ .

Lattice structure of Rickart \*-rings [i.e., Foulis semigroups that happen to be an involution ring] under this order has been studied by M.F.Janowitz in 1983, and later by J.Cīrulis in 2015.

• Most of the results can be transferred to Foulis semigroups.

It turns out that presence of involution is not necessary for the coordinatization theorem: a class of ordinary Baer semigroups do the job.

Involution is not necessary also to characterize the star order on a Foulis semigroup.

#### **Definition** [M.F.Janowitz (1965)]

A Baer semigroup is an algebra  $(S, \cdot, 0, \cdot, \prime)$  such that

- $(S, \cdot, 0)$  is a semigroup with zero,
- x' and x' are idempotents,
- the left and right annihilator axioms are fulfilled:

yx = 0 iff yx' = y, xy = 0 iff x'y = y.

Both annihilator axioms may be replaced by the respective Adams identities

$$(yx)'yx' = (yx)'y, \quad x'y(xy)' = y(xy)'.$$

Again,  $1 := 0^{1} = 0^{1}$  is the unity in S.

The ranges of the operations ' and ' [now not unique!] still coincide. Let P stand for the common range; call idemotents in P closed.

#### Definition

A Baer semigroup is said to be *strong* if, for all  $p, q \in P$ , • p' = p', •  $pq \in P$  iff  $qp \in P$ .

#### Example

The  $(\cdot, 0, \cdot, \prime)$ -subreduct of a Foulis semigroup is a strong Baer semigroup (with its closed projections in the role of closed idempotents).

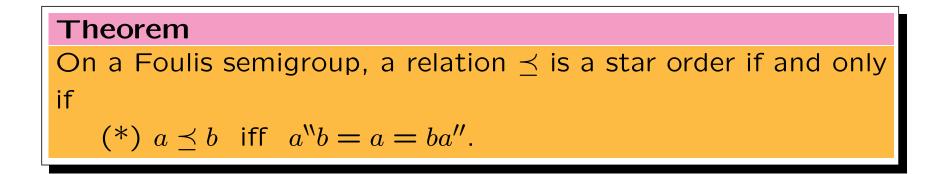
#### Theorem

(a) In a strong Baer semigroup, the closed idempotents still form an orthomodular lattice with the ordering given by

 $e \leq f$  iff ef = e [iff fe = e].

(b) every OM lattice is isomorphic to the lattice of closed idempotents of a strong Baer semigroup.

#### 2. STAR ORDER ON STRONG BAER SEMIGROUPS

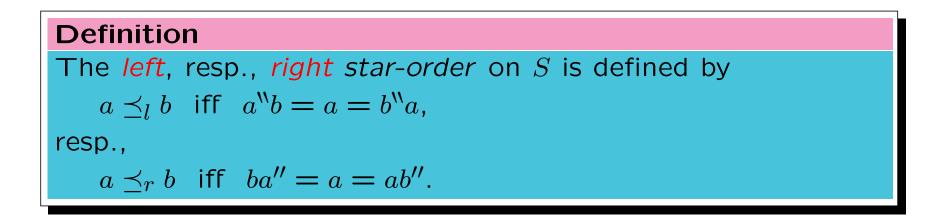


#### Definition

The star order  $\leq$  on a strong Baer semigroup is defined by the condition (\*).

In the rest, let S be a strong Baer semigroup.

Star order:  $a \leq b$  iff  $a^{"}b = a = ba''$ .



#### For all $a, b, a \leq b$ iff $a \leq_l b$ and $a \leq_r b$ .

[left/right Baer semigroups].

Proposition (for star order)
In S,
(a) 0 is the least element,
(b) $P = [0, 1],$
(c) the order $\leq$ agrees on P with the natural ordering of closed
idempotents; in particular,
(d) meets and joins in $P$ agree with those existing in $S$ ,
(e) S has the greatest element only if $S = P$ .

The proposition holds true for left/right star-ordered strong Baer semigroups.

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Theorem (for star order)
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(a) Every initial segment of S is an orthomodular lattice, in which joins and meets agree with those existing on the whole S.

(b) Any segment [0, x] of S is embedded into the sublattices  $[0, x^{"}]$  and [0, x''] of P.

Again, the proposition holds true for left/right star-ordered strong Baer semigroups. Even more, the segment  $[0, x]_l$ , resp.,  $[0, x]_r$ , is isomorphic to  $[0, x^{\text{W}}]$ , resp., [0, x''].

Recall that  $[0, x] = [0, x]_l \cap [0, x]_r$ .

A subset of S is said to be *compatible* if any pair of its elements has an upper bound.

P is an example of a maximal compatible subset of S.

## Theorem Let *M* be a maximal compatible subset of *S*. Then (a) *M* is a lattice isomorphic to an ideal of *P*, (b) if *M* has a greatest element which is invertible, then the ideal coincides with *P*.

Likewise for one-sided star orders and one-side [opposite!] in-vertibility.

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