Lattices with unique complementation

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The famous Ogasawara-Sasaki Theorem [3], see also Birkhoff-Ward [1], states that every uniquely complemented atomic lattice is distributive. The conjecture that every uniquely complemented lattice is distributive was rejected by V.N. Salij in 1972, see [4]. He constructed an infinite non-atomic uniquely complemented lattice which is not distributive. Several results concerning distributivity of uniquely complemented lattices are collected in [2] and [4].

Our aim is to get two identities in two variables which ensure that a lattice with complementation satisfying these identities is distributive. For the proof of this theorem it can be used the result from [3] but our proof is short and clear and hence we prefer to present it without application of known results of other authors. Moreover, we show an easy computation which derive the distributivity law directly from our identities.

By a *lattice with complementation* we mean a bounded lattice $\mathcal{L} = (L; \vee, \wedge, ', 0, 1)$ where for each $x \in L$ the element x' is a *complement* of x, i.e. $x \vee x' = 1$ and $x \wedge x' = 0$. By a *uniquely complemented lattice* we mean a bounded lattice where every element has just one complement; hence it is a lattice with complementation where $x \vee y = 1$ and $x \wedge y = 0$ yield $y = x'$.

Consider the following two identities

$$x' \vee (x \wedge y) = x' \vee y \qquad \text{and} \qquad x' \wedge (x \vee y) = x' \wedge y \tag{1}$$

Denote by $\mathcal{W}$ the variety of lattices with complementation satisfying (1).

At first, we prove the following lemma.

Lemma 1

Every lattice with complementation satisfying (1) *is uniquely complemented.*

Proof.

Let $x \in L$ and assume that $z \in L$ is a complement of x. By (1) we obtain

$$x' = x' \vee 0 = x' \vee (x \wedge z) = x' \vee z \qquad \text{thus} \qquad z \leq x'$$

$$x' = x' \wedge 1 = x' \wedge (x \vee z) = x' \wedge z \qquad \text{thus} \qquad x' \leq z.$$

Hence $z = x'$ proving the assertion. □

By Lemma 1, every lattice with complementation satisfying (1) satisfies also $x'' = x$. Hence, it satisfies also the dual identities

$$x \vee (x' \wedge y) = x \vee y \qquad \text{and} \qquad x \wedge (x' \vee y) = x \wedge y \tag{2}$$

which are in fact equivalent to (1).

A lattice with complementation is called an *orthocomplemented lattice* if $x'' = x$ and $x \leq y$ implies $y' \leq x'$.

Lemma 2

Let $\mathcal{L}$ be a lattice with complementation satisfying (1). Then $\mathcal{L}$ is an orthocomplemented lattice.

Proof.

By Lemma 1 we know that $x'' = x$, Using of (1), we compute

$$(x \wedge y) \vee (x' \vee y') = ((x \wedge y) \vee x') \vee y' = x' \vee y \vee y' = 1$$

and

$$(x \wedge y) \wedge (x' \vee y') = x \wedge (y \wedge (x' \vee y') = x \wedge y \wedge x' = 0.$$

Hence, $x' \vee y'$ is a complement of $x \wedge y$. By Lemma 1, the complementation is unique and thus

$$x' \vee y' = (x \wedge y)'.$$

Taking $x \leq y$ we infer $x' \vee y' = (x \wedge y)' = x'$ where $y' \leq x'$. □

Now, we can prove our main result.

Theorem 1

The variety $\mathcal{W}$ is the variety of Boolean algebras.

Proof.

If $\mathcal{L} = (L; \vee, \wedge,', 0, 1)$ is a Boolean algebra then, using distributivity, we immediately obtain the identities (1).
Conversely, to prove that every $\mathcal{L} \in \mathcal{W}$ is a Boolean algebra we need only to show that $\mathcal{L}$ is distributive. For this, consider the free algebra $F_w(x, y, z) \in \mathcal{W}$ generated by three free generators x, y, z.
Evidently, this complemented lattice is atomic, its atoms are $x \wedge y \wedge z,\ x \wedge y \wedge z',\ x \wedge y' \wedge z, \dots, x' \wedge y' \wedge z'$. By Birkhoff-Ward Theorem, every atomic uniquely complemented lattice is distributive and hence in $F_w(x, y, z)$ it holds

$$z \wedge (x \vee y) = (z \wedge x) \vee (z \wedge y).$$

Since this holds in a free lattice of $\mathcal{W}$, it holds in every $\mathcal{L} \in \mathcal{W}$. □

In what follows, we show that distributivity can be derived from the identities (1) directly, i.e. we can prove that a lattice with complementation is distributive of and only if it satisfies (1). Hence, we need not in fact to investigate the whole variety $\mathcal{W}$ as in the previous theorem.

Derivation of distributivity from the identities (1).
Using associativity and (2), we compute

$$x \vee ((x' \wedge y) \vee z) = (x \vee (x' \wedge y)) \vee z = x \vee y \vee z, \qquad (3)$$

applying this and (2), we obtain

$$x \wedge (y \vee z) = x \wedge (x' \vee y \vee z) = x \wedge (x' \vee ((x \wedge y) \vee z) = \qquad (4)$$

$$= x \wedge ((x \wedge y) \vee z) = x \wedge (z \vee (x \wedge y)).$$

Since lattice identities as well as (2) are dual, we obtain the following by dualization of (3):

$$x \vee (y \wedge z) = x \vee (z \wedge (x \vee y)). \tag{5}$$

Since $x \vee y \geq x$ and $x \vee y \geq (x \vee y) \wedge z$, we obtain immediately

$$(x \vee y) \wedge (x \vee ((x \vee y) \wedge z)) = x \vee ((x \vee y) \wedge z). \tag{6}$$

Applying (6) and (5), we derive

$$y \vee ((x \vee y) \wedge z) = (x \vee y) \wedge (y \vee ((x \vee y) \wedge z)) = (x \vee y) \wedge (y \vee z). \tag{7}$$

From (5) and (7) we finally obtain

$$y \vee (x \wedge z) = y \vee (z \wedge (x \vee y)) = (x \vee y) \wedge (y \vee z) = (y \vee x) \wedge (y \vee z)$$

which is the distributivity law.

The end!

Thanks for your attention!!

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