TECHNISCHE UNIVERSITÄT WIEN

Reconstructing the topology on monoids and clones of the rationals

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[2] Supported by CONACYT.

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Topology on transformation monoids and clones

A a set, $n \in \mathbb{N}$.

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Transformation monoids

$F \subseteq A^A$, i.e. $n = 1$ induced subspace topology

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- H (BPP) Horn clone

New stuff

Our contribution…

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Now let's prove this.

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A helpful and inspiring lemma

Bodirsky, Pinsker, Pongrácz, Lemma 12

- $M \leq \mathrm{O}_A^{(1)}$ closed submonoid ($|A| = \aleph_0$),
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We prove for any injective hom $\xi\colon M \hookrightarrow \boldsymbol{E}$:

$(\forall g \in G : \xi(g) = g) \implies (\forall f \in M : \xi(f) = f)$

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$\Longrightarrow \quad \xi(f) = f \qquad$ (for $g \in \Psi \subseteq M$ is injective)

On the way to automatic homeomorphicity for $E = \operatorname{End}(\mathbb{Q}, \leq)$

We know...

... $M = \operatorname{End}(\mathbb{Q}, <)$ has automatic homeomorphicity by [Bodirsky, Pinsker, Pongrácz, Lemma 12].

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Observations

- $f \in E$ surjective $\implies$
  $\forall g \in \mathbb{Q}^{\mathbb{Q}}: \quad f \circ g = \mathrm{id}_{\mathbb{Q}} \implies g \in M$
  i.e. right-inverse maps are embeddings

$\implies$ surjective $f \in E$ are characterizable by their right-inverses.

- trickery $\implies \forall h \in E\, \exists f \in E$ surj $\exists g \in M: \quad h = f \circ g$

Variation upon a lemma by BPP

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Variation upon a lemma by BPP

Very slightly differing from Lemma 12

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All orbits: $(\Omega_i)_{i \in I}$

$\forall i \in I: \qquad \Omega_i = \left\{ a_B^i \;\middle|\; B \in [\mathbb{Q}]^{n_i} \right\} \qquad$ (rank $n_i \in \mathbb{N}$)

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We prove continuity and openness of $\theta\colon E \longrightarrow E'$.

Automatic homeomorphicity of $\mathrm{Pol}\,(\mathbb{Q}, \leq)$

Method

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continuity use idempotents constructed for E
\+ method above