On CSP Dichotomy Conjecture

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Outline

Let G be a finite set of predicates on a finite set A.

CSP(G)

Given: a conjunction of predicates, i.e. a formula

$$\rho_1(x_{i_{1,1}},\dots,x_{i_{1,n_1}})\wedge\dots\wedge\rho_s(x_{i_{s,1}},\dots,x_{i_{s,n_s}}),$$

where $\rho_1,\dots,\rho_s\in G$.
Decide: whether the formula is satisfiable.

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Example

$A = \{0, 1, 2\}, G = \{x < y, x \leq y\}$.
CSP instances:
$x_1 < x_2 \wedge x_2 < x_3 \wedge x_3 < x_4$,

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$x_1 \leq x_2 \wedge x_2 \leq x_3 \wedge x_3 \leq x_1$, $x_1 = x_2 = x_3 = 0$.

A weak near unanimity operation (WNU) is an operation f satisfying
$f(x, x, \ldots, x) = x$ and
$f(x, \ldots, x, y) = f(x, \ldots, x, y, x) = \cdots = f(y, x, \ldots, x)$.

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Theorem[Ralph McKenzie and Miklós Maróti]

CSP(G) is NP-complete if no WNU preserving G.

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The conjecture was proved

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Absorption

A subuniverse B absorbs A if there exists an operation $f \in \mathrm{Clo}(w)$ such that $f(B, \dots, B, A, B, \dots, B) \subseteq B$ for any position of A.

- If f is binary, then the absorption is called binary.

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Subdirect

A relation $\rho \subseteq A_1 \times \cdots \times A_n$ is called subdirect if $\mathrm{pr}_i(\rho) = A_i$ for every i.

A WNU w is called minimal if there doesn't exist WNU w' such that $\mathrm{Clo}(w') \subsetneq \mathrm{Clo}(w)$.

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An operation f is called cyclic if f is idempotent and

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Theorem [L.Barto, M. Kozik, 2012]

Let $\mathcal{V}$ be an idempotent variety generated by a finite algebra $\mathbf{A}$ then the following are equivalent.

- $\mathcal{V}$ is a Taylor variety;
- $\mathcal{V}$ (equivalently the algebra $\mathbf{A}$) has a cyclic term;
- $\mathcal{V}$ (equivalently the algebra $\mathbf{A}$) has a cyclic term of arity p, for every prime $p > |A|$.

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Corollary 1

For every WNU w there exists a cyclic operation $w' \in \mathrm{Clo}(w)$ of arity at most $2|A|$ (which is also a WNU).

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- It is sufficient to prove CSP Dichotomy Conjecture just for minimal WNU.

Why minimal WNU?

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Lemma

Suppose B absorbs A with a binary operation $f \in \mathrm{Clo}(w)$, w is a minimal WNU.
Then $w(A, \ldots, A, B, A, \ldots, A) \subseteq B$ for any position of B.

Bijective WNU

A WNU w is called bijective if for any two tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ that differ just in one component we have $w(a_1, \ldots, a_n) \neq w(b_1, \ldots, b_n)$.

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Equivalent definition of a bijective WNU

A WNU w is called bijective if for any i and any tuple $(a_1, \ldots, a_n)$ the operation $h(x) = w(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$ is bijective.

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Example 1: Quasi-linear WNU

$w(x_1, \ldots, x_n) = x_1 + x_2 + \ldots + x_n$ where $(A; +)$ is an abelian group

Example 2: A bijective WNU that is not Abelian.

Define a Mal'tsev operation and a WNU on $\mathbb{Z}_2 \times \mathbb{Z}_2$.

$$m^{(1)}(x, y, z) = x^{(1)} + y^{(1)} + z^{(1)}.$$

$$m^{(2)}(x, y, z) = x^{(2)} + y^{(2)} + z^{(2)} + x^{(1)} z^{(1)} (y^{(1)} + 1).$$

$$w(x_1, x_2, x_3, x_4, x_5) = m(m(x_1, x_2, x_3), x_2, m(x_4, x_2, x_5)).$$

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- $w(x, \ldots, x, y) = w(x, \ldots, x, y, x) = \cdots = w(y, x, \ldots, x) = y$.
- the WNU w is a minimal WNU.

Fact

Suppose σ_1 and σ_2 are congruences on A, w/σ_1 and w/σ_2 are bijective. Then $w/(\sigma_1 \cap \sigma_2)$ is bijective.

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Lemma

Suppose $\rho \subseteq A_1 \times A_2$ is subdirect, the WNU w is bijective on A_2, no binary absorption on A_1, then $\rho = (B_1 \times C_1) \cup \cdots \cup (B_s \times C_s)$ where $A_1 = B_1 \sqcup \cdots \sqcup B_s$, $A_2 = C_1 \sqcup \cdots \sqcup C_s$.

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- No binary absorption on $D_1, \ldots, D_n$.
- Let σ_i be the minimal congruence on D_i such that w/σ_i is bijective.
- Factorize all the constraints, i.e. replace every predicate ρ by

$$\rho'(x_1, \ldots, x_n) = \exists y_1 \ldots \exists y_n \ \rho(y_1, \ldots, y_n) \wedge (x_1, y_1) \in \sigma_{i_1} \wedge \cdots \wedge (x_n, y_n) \in \sigma_{i_n}$$

The obtained CSP instance we denote by Θ^F

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- Let $(S_1, \ldots, S_n)$ be a solution of Θ^F.

then the restriction of Θ to $(S_1, \ldots, S_n)$ is 1-consistent.

We say that a relation $\rho \in A_1 \times \cdots \times A_n$ is compatible with a congruence σ on A_i if

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- Can we avoid the condition $\mathrm{pr}_{1,2,\ldots,n-1}(\rho) = A_1 \times \cdots \times A_{n-1}$?

Definitions

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- Every relation can be represented as a conjunction of critical relations.
- A subset $C \subsetneq A$ is called a center on A if there exists a subdirect binary relation $\rho \subseteq A \times A$ such that $\rho \in \mathrm{Inv}(w)$ and $C = \{c \in A \mid \forall d \in A\colon (c, d) \in \rho\}$.

Conjecture 1

Suppose $\rho \subseteq A_1 \times A_2 \times A_3$ is a critical subdirect relation in $\mathrm{Inv}(w)$, and

1. w is a minimal WNU.
2. ρ is compatible with σ_3.
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CSP Dichotomy Conjecture holds if $|A| \leq 5$.

What I need to prove

Conjecture 2

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4. no binary absorption or center on A_1 and A_2
5. w/κ is quasi-linear for every i and every maximal congruence κ on A_i.
6. There exists $c \in A_3$ such that $(\forall a_1 \in A_1 \exists a_2 \in A_2 : (a_1, a_2, c) \in \rho)$ and $(\forall a_2 \in A_2 \exists a_1 \in A_1 : (a_1, a_2, c) \in \rho)$

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Thank you for your attention

Algorithm

1-3 Preliminary steps. We repeat them if necessary.

4 If all domains are unary, we get a solution.

5 If there exists a binary absorbtion: Apply Absorbing Reduction, provide 1-consistency, and go to Step 4.

6 If there exists a center: Apply Central Reduction, provide 1-consistency, and go to Step 4.

7 If we get all functions after factorization: Apply "All Functions" reduction, provide 1-consistency, and go to Step 4.

8 If the WNU w is bijective after factorization

1. solve the factorized CSP.
2. if it has a solution, apply Linear Reduction and go to Step 4.
3. if we can remove a constraint or split a variable to get a CSP instance Ω such that Ω^F has no solutions, we do this while possible.
4. if we can remove a constraint or split a variable to get a CSP instance Ω such that Ω^F has a solution $(S_1, \ldots, S_n)$ but the reduction of Ω to $(S_1, \ldots, S_n)$ has no solutions, then we consider the reduction and go to Step 5.
5. if Θ^F has no solutions and we cannot remove or split, then we reduce the original domain A_i to A_i'.