On CSP Dichotomy Conjecture

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Outline

1. What is CSP?
2. CSP Dichotomy Conjecture
3. Minimal WNU
4. Bijective WNU
5. Main Conjecture
Let $G$ be a finite set of predicates on a finite set $A$.

**CSP($G$)**

Given: a conjunction of predicates, i.e. a formula

$$\rho_1(x_{i_1,1}, \ldots, x_{i_1,n_1}) \land \cdots \land \rho_s(x_{i_s,1}, \ldots, x_{i_s,n_s}),$$

where $\rho_1, \ldots, \rho_s \in G$.

Decide: whether the formula is satisfiable.
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Decide: whether the formula is satisfiable.

**Example**

$A = \{0, 1, 2\}$, $G = \{x < y, x \leq y\}$.

CSP instances:

$x_1 < x_2 \land x_2 < x_3 \land x_3 < x_4,$
Constraint Satisfaction Problem

Let $G$ be a finite set of predicates on a finite set $A$.

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CSP instances:
\begin{align*}
&x_1 < x_2 \land x_2 < x_3 \land x_3 < x_4, \text{ No solutions} \\
&x_1 \leq x_2 \land x_2 \leq x_3 \land x_3 \leq x_1,
\end{align*}
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\[ \rho_1(x_{i_1,1}, \ldots, x_{i_1,n_1}) \land \cdots \land \rho_s(x_{i_s,1}, \ldots, x_{i_s,n_s}), \]

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$A = \{0, 1, 2\}$, $G = \{x < y, x \leq y\}$.

CSP instances:

- $x_1 < x_2 \land x_2 < x_3 \land x_3 < x_4$, No solutions
- $x_1 \leq x_2 \land x_2 \leq x_3 \land x_3 \leq x_1$, $x_1 = x_2 = x_3 = 0$. 
A weak near unanimity operation (WNU) is an operation $f$ satisfying

$$
\begin{align*}
    f(x, x, \ldots, x) &= x \\
    f(x, \ldots, x, y) &= f(x, \ldots, x, y, x) = \cdots = f(y, x, \ldots, x).
\end{align*}
$$

Suppose $(x = c)$ belongs to $G$ for every $c \in A$. Only idempotent case!

**Conjecture**

CSP($G$) is solvable in polynomial time if there exists a WNU preserving $G$. CSP($G$) is NP-complete otherwise.

Theorem

[Ralph McKenzie and Miklós Maróti]

CSP($G$) is NP-complete if no WNU preserving $G$. [Dmitriy Zhuk](zhuk.dmitriy@gmail.com) (Moscow State University)

On CSP Dichotomy Conjecture
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Clone generated by an operation

For an operation \( f \) by \( \text{Clo}(f) \) we denote the clone generated by \( f \).
Definitions

Clone generated by an operation

For an operation $f$ by $\text{Clo}(f)$ we denote the clone generated by $f$.

Absorption

A subuniverse $B$ absorbs $A$ if there exists an operation $f \in \text{Clo}(w)$ such that $f(B, \ldots, B, A, B, \ldots, B) \subseteq B$ for any position of $A$.

- If $f$ is binary, then the absorption is called binary.
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1-consistency

A CSP instance is called 1-consistent if every \( x_i \) in any constraint takes all values from the domain of \( x_i \).
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Subdirect
A relation \( \rho \subseteq A_1 \times \cdots \times A_n \) is called subdirect if \( \text{pr}_i(\rho) = A_i \) for every \( i \).
A WNU $w$ is called **minimal** if there doesn’t exist WNU $w'$ such that $\text{Clo}(w') \nsubseteq \text{Clo}(w)$.
A WNU $w$ is called \textit{minimal} if there doesn’t exist WNU $w'$ such that $\text{Clo}(w') \subsetneq \text{Clo}(w)$.

An operation $f$ is called \textit{cyclic} if $f$ is idempotent and

$$f(x_1, x_2, \ldots, x_n) = f(x_2, x_3, \ldots, x_n, x_1).$$
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**Theorem [L. Barto, M. Kozik, 2012]**

Let $\mathcal{V}$ be an idempotent variety generated by a finite algebra $A$ then the following are equivalent.

- $\mathcal{V}$ is a Taylor variety;
- $\mathcal{V}$ (equivalently the algebra $A$) has a cyclic term;
- $\mathcal{V}$ (equivalently the algebra $A$) has a cyclic term of arity $p$, for every prime $p > |A|$. 

**Corollary 1**

For every WNU $w$ there exists a cyclic operation $w' \in \text{Clo}(w)$ of arity at most $2|A|$ (which is also a WNU).
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**Corollary 2**

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A WNU \( w \) is called \textbf{minimal} if there doesn’t exist WNU \( w' \) such that \( \text{Clo}(w') \subsetneq \text{Clo}(w) \).

An operation \( f \) is called \textbf{cyclic} if \( f \) is idempotent and

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\textbf{Corollary 2}

For every WNU \( w \) there exists a minimal WNU \( w' \in \text{Clo}(w) \)

- It is sufficient to prove CSP Dichotomy Conjecture just for minimal WNU.
Why minimal WNU?

Lemma

Suppose $B$ absorbs $A$ with a binary operation $f \in \text{Clo}(w)$, $w$ is a minimal WNU. Then $w(A, \ldots, A, B, A, \ldots, A) \subseteq B$ for any position of $B$. 

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On CSP Dichotomy Conjecture
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A WNU $w$ is called bijective if for any two tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ that differ just in one component we have

$w(a_1, \ldots, a_n) \neq w(b_1, \ldots, b_n)$. 
**Bijective WNU**

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**Equivalent definition of a bijective WNU**

A WNU $w$ is called **bijective** if for any $i$ and any tuple $(a_1, \ldots, a_n)$ the operation $h(x) = w(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$ is bijective.
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**Example 1: Quasi-linear WNU**

$w(x_1, \ldots, x_n) = x_1 + x_2 + \ldots + x_n$ where $(A; +)$ is an abelian group
Example 2: A bijective WNU that is not Abelian.

Define a Mal’tsev operation and a WNU on \( \mathbb{Z}_2 \times \mathbb{Z}_2 \).

\[
\begin{align*}
m^{(1)}(x, y, z) &= x^{(1)} + y^{(1)} + z^{(1)}. \\
m^{(2)}(x, y, z) &= x^{(2)} + y^{(2)} + z^{(2)} + x^{(1)}z^{(1)}(y^{(1)} + 1). \\
w(x_1, x_2, x_3, x_4, x_5) &= m(m(x_1, x_2, x_3), x_2, m(x_4, x_2, x_5)).
\end{align*}
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$$w(x_1, x_2, x_3, x_4, x_5) = m(m(x_1, x_2, x_3), x_2, m(x_4, x_2, x_5)).$$

$$w(x, \ldots, x, y) = w(x, \ldots, x, y, x) = \cdots = w(y, x, \ldots, x) = y.$$
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w(x_1, x_2, x_3, x_4, x_5) = m(m(x_1, x_2, x_3), x_2, m(x_4, x_2, x_5)).
\]

- \( w(x, \ldots, x, y) = w(x, \ldots, x, y, x) = \cdots = w(y, x, \ldots, x) = y. \)
- the WNU \( w \) is a minimal WNU.
Motivation

Fact
Suppose $\sigma_1$ and $\sigma_2$ are congruences on $A$, $w/\sigma_1$ and $w/\sigma_2$ are bijective. Then $w/(\sigma_1 \cap \sigma_2)$ is bijective.
Motivation

Fact

Suppose $\sigma_1$ and $\sigma_2$ are congruences on $A$, $w/\sigma_1$ and $w/\sigma_2$ are bijective. Then $w/(\sigma_1 \cap \sigma_2)$ is bijective.

Let $\sigma_i$ be the minimal congruence on $A_i$ such that $w/\sigma_i$ is bijective.
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Fact
Suppose $\sigma_1$ and $\sigma_2$ are congruences on $A$, $w/\sigma_1$ and $w/\sigma_2$ are bijective. Then $w/(\sigma_1 \cap \sigma_2)$ is bijective.

Let $\sigma_i$ be the minimal congruence on $A_i$ such that $w/\sigma_i$ is bijective.

Lemma
Suppose $\rho \subseteq A_1 \times A_2$ is subdirect, the WNU $w$ is bijective on $A_2$, no binary absorption on $A_1$, then $\rho = (B_1 \times C_1) \cup \cdots \cup (B_s \times C_s)$ where $A_1 = B_1 \sqcup \cdots \sqcup B_s$, $A_2 = C_1 \sqcup \cdots \sqcup C_s$. 
Motivation

Corollary

Suppose

- We have a 1-consistent CSP instance $\Theta$ with domains $D_1, \ldots, D_n$. 
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- We have a 1-consistent CSP instance $\Theta$ with domains $D_1, \ldots, D_n$.
- No binary absorption on $D_1, \ldots, D_n$.
- Let $\sigma_i$ be the minimal congruence on $D_i$ such that $w/\sigma_i$ is bijective.
Motivation

Corollary

Suppose
- We have a 1-consistent CSP instance $\Theta$ with domains $D_1, \ldots, D_n$.
- No binary absorption on $D_1, \ldots, D_n$.
- Let $\sigma_i$ be the minimal congruence on $D_i$ such that $w/\sigma_i$ is bijective.
- Factorize all the constraints, i.e. replace every predicate $\rho$ by $\rho'(x_1, \ldots, x_n) = \exists y_1 \ldots \exists y_n \rho(y_1, \ldots, y_n) \land (x_1, y_1) \in \sigma_i \land \cdots \land (x_n, y_n) \in \sigma_i$

The obtained CSP instance we denote by $\Theta^F$
Corollary

Suppose

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- Factorize all the constraints, i.e. replace every predicate $\rho$ by
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  The obtained CSP instance we denote by $\Theta^F$
- Let $(S_1, \ldots, S_n)$ be a solution of $\Theta^F$.

then the restriction of $\Theta$ to $(S_1, \ldots, S_n)$ is 1-consistent.
We say that a relation \( \rho \in A_1 \times \cdots \times A_n \) is compatible with a congruence \( \sigma \) on \( A_i \) if

\[(a_1, \ldots, a_n) \in \rho \land (a_i, b) \in \sigma \Rightarrow (a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \in \rho.\]
We say that a relation $\rho \in A_1 \times \cdots \times A_n$ is compatible with a congruence $\sigma$ on $A_i$ if

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Let $\sigma_i$ be the minimal congruence on $A_i$ such that $w/\sigma_i$ is bijective.

What I need to prove CSP Dichotomy Conjecture
We say that a relation $\rho \in A_1 \times \cdots \times A_n$ is compatible with a congruence $\sigma$ on $A_i$ if

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Let $\sigma_i$ be the minimal congruence on $A_i$ such that $w/\sigma_i$ is bijective.

**What I need to prove CSP Dichotomy Conjecture**

Suppose $\rho \in A_1 \times \cdots \times A_n$, $\rho$ is compatible with $\sigma_j$. Then $\rho$ is compatible with $\sigma_i$ for every $i$. 
We say that a relation \( \rho \in A_1 \times \cdots \times A_n \) is compatible with a congruence \( \sigma \) on \( A_i \) if

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(a_1, \ldots, a_n) \in \rho \land (a_i, b) \in \sigma \Rightarrow (a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \in \rho.
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Let \( \sigma_i \) be the minimal congruence on \( A_i \) such that \( w/\sigma_i \) is bijective.

**What I need to prove CSP Dichotomy Conjecture**

Suppose \( \rho \in A_1 \times \cdots \times A_n \), \( \rho \) is compatible with \( \sigma_j \). Then \( \rho \) is compatible with \( \sigma_i \) for every \( i \).

**Lemma**

Suppose \( \rho \subseteq A_1 \times A_2 \times \cdots \times A_n \) is preserved by a WNU \( w \), \( \rho \) is compatible with \( \sigma_n \), \( \text{pr}_{1,2,\ldots,n-1}(\rho) = A_1 \times \cdots \times A_{n-1} \), no binary absorption on \( A_1, A_2, \ldots, A_n \). Then \( \rho \) is compatible with \( \sigma_i \) for every \( i \).
We say that a relation $\rho \in A_1 \times \cdots \times A_n$ is compatible with a congruence $\sigma$ on $A_i$ if

$$(a_1, \ldots, a_n) \in \rho \land (a_i, b) \in \sigma \implies (a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \in \rho.$$ 

Let $\sigma_i$ be the minimal congruence on $A_i$ such that $w/\sigma_i$ is bijective.

**What I need to prove CSP Dichotomy Conjecture**

Suppose $\rho \in A_1 \times \cdots \times A_n$, $\rho$ is compatible with $\sigma_j$. Then $\rho$ is compatible with $\sigma_i$ for every $i$.

**Lemma**

Suppose $\rho \subseteq A_1 \times A_2 \times \cdots \times A_n$ is preserved by a WNU $w$, $\rho$ is compatible with $\sigma_n$, $\text{pr}_{1,2,\ldots,n-1}(\rho) = A_1 \times \cdots \times A_{n-1}$, no binary absorption on $A_1, A_2, \ldots, A_n$. Then $\rho$ is compatible with $\sigma_i$ for every $i$.

Can we avoid the condition $\text{pr}_{1,2,\ldots,n-1}(\rho) = A_1 \times \cdots \times A_{n-1}$?
Definitions

- By $\text{Inv}(f)$ we denote all invariants of $f$. 
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- A relation is called **critical in a relational clone $C$** if it is completely $\cap$-irreducible in $C$ and directly indecomposable (Keith A.Kearnes and Ágnes Szendrei).

Equiv alently, a relation $\rho$ is critical in a relational clone $C$ if it cannot be represented as $\rho_1(\ldots) \wedge \cdots \wedge \rho_n(\ldots)$, where for every $i$ we have $\rho_i \in C$, $\text{arity}(\rho) > \text{arity}(\rho_i)$ or $\rho \not\subseteq \rho_i$.

Every relation can be represented as a conjunction of critical relations.

A subset $C \subseteq A$ is called a center on $A$ if there exists a sub direct binary relation $\rho \subseteq A \times A$ such that $\rho \in \text{Inv}(w)$ and $C = \{c \in A | \forall d \in A: (c, d) \in \rho\}$.
Definitions

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Definitions

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- A subset $C \subsetneq A$ is called a **center on $A$** if there exists a subdirect binary relation $\rho \subseteq A \times A$ such that $\rho \in \text{Inv}(w)$ and $C = \{ c \in A \mid \forall d \in A: (c, d) \in \rho \}$. 
Conjecture 1

Suppose $\rho \subseteq A_1 \times A_2 \times A_3$ is a critical subdirect relation in $\text{Inv}(w)$, and

1. $w$ is a minimal WNU.
2. $\rho$ is compatible with $\sigma_3$.
3. (parallelogram property) $\forall a_1, a_2, a_3, b_1, b_2, b_3$:
   $$(a_1, a_2, b_3), (b_1, b_2, a_3), (b_1, b_2, b_3) \in \rho \Rightarrow (a_1, a_2, a_3) \in \rho.$$ 

Then $\rho$ is compatible with $\sigma_1$ and $\sigma_2$. 

Theorem

Conjecture 1 $\Rightarrow$ CSP Dichotomy Conjecture.

Theorem

Conjecture 1 holds if $|A_i| \leq 5$ for every $i$.

Corollary

CSP Dichotomy Conjecture holds if $|A| \leq 5$. 

Dmitriy Zhuk zhuk.dmitriy@gmail.com (Moscow State University)
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Then $\rho$ is compatible with $\sigma_1$ and $\sigma_2$, or, equivalently,

$(a_1, a_2, a_3) \in \rho, (a_1, b_1) \in \sigma_1, (a_2, b_2) \in \sigma_2, (a_3, b_3) \in \sigma_3 \Rightarrow (b_1, b_2, b_3) \in \rho$.
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Theorem

Conjecture 1 $\Rightarrow$ CSP Dichotomy Conjecture.

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Conjecture 1 holds if $|A_i| \leq 5$ for every $i$.

Corollary

CSP Dichotomy Conjecture holds if $|A| \leq 5$. 
Conjecture 2

Suppose $\rho \subseteq A_1 \times A_2 \times A_3$ is a critical subdirect relation in $\text{Inv}(w)$, and

1. $w$ is a minimal WNU.
2. $\rho$ is compatible with $\sigma_3$.
3. (parallelogram property) $\forall a_1, a_2, a_3, b_1, b_2, b_3 : (a_1, a_2, b_3), (b_1, b_2, a_3), (b_1, b_2, b_3) \in \rho \Rightarrow (a_1, a_2, a_3) \in \rho$.
4. no binary absorption or center on $A_1$ and $A_2$.
5. $w/\kappa$ is quasi-linear for every $i$ and every maximal congruence $\kappa$ on $A_i$.
6. There exists $c \in A_3$ such that $(\forall a_1 \in A_1 \exists a_2 \in A_2 : (a_1, a_2, c) \in \rho)$ and $(\forall a_2 \in A_2 \exists a_1 \in A_1 : (a_1, a_2, c) \in \rho)$.

Then $\rho$ is compatible with $\sigma_1$ and $\sigma_2$.
Conjecture 2

Suppose $\rho \subseteq A_1 \times A_2 \times A_3$ is a critical subdirect relation in $\text{Inv}(w)$, and

1. $w$ is a minimal WNU.
2. $\rho$ is compatible with $\sigma_3$.
3. (parallelogram property) $\forall a_1, a_2, a_3, b_1, b_2, b_3:\ (a_1, a_2, b_3), (b_1, b_2, a_3), (b_1, b_2, b_3) \in \rho \Rightarrow (a_1, a_2, a_3) \in \rho$.
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Then $\rho$ is compatible with $\sigma_1$ and $\sigma_2$, or, equivalently,

$\forall a_1, a_2, a_3, b_1, b_2, b_3:\ (a_1, a_2, a_3) \in \rho, (a_1, b_1) \in \sigma_1, (a_2, b_2) \in \sigma_2, (a_3, b_3) \in \sigma_3 \Rightarrow (b_1, b_2, b_3) \in \rho$. 
Theorem

Conjecture 2 $\Rightarrow$ CSP Dichotomy Conjecture.

Corollary

CSP Dichotomy Conjecture holds if $|A| \leq 7$.
Theorem
Conjecture 2 $\Rightarrow$ CSP Dichotomy Conjecture.

Theorem
Conjecture 2 holds if $|A_i| \leq 7$ for every $i$. 
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Conjecture 2 holds if $|A_i| \leq 7$ for every $i$.

Corollary
CSP Dichotomy Conjecture holds if $|A| \leq 7$. 
I need your help with Conjecture 1.

Suppose \( \rho \subseteq A_1 \times A_2 \times A_3 \) is a critical sub direct relation in \( \text{Inv}(w) \), and \( w \) is a minimal WNU. \( \rho \) is compatible with \( \sigma_3 \). (parallelogram property) 

\[ \forall a_1, a_2, a_3, b_1, b_2, b_3 : (a_1, a_2, b_3), (b_1, b_2, a_3), (b_1, b_2, b_3) \in \rho \Rightarrow (a_1, a_2, a_3) \in \rho. \]

Then \( \rho \) is compatible with \( \sigma_1 \) and \( \sigma_2 \), or, equivalently, 

\[ (a_1, a_2, a_3) \in \rho, (a_1, b_1, a_2) \in \sigma_1, (a_2, b_2, a_3) \in \sigma_2, (a_3, b_3, a_3) \Rightarrow (b_1, b_2, b_3) \in \rho. \]
I need your help with Conjecture 1.

Let $\sigma_i$ be the minimal congruence on $A_i$ such that $w/\sigma_i$ is bijective.

**Conjecture 1**

Suppose $\rho \subseteq A_1 \times A_2 \times A_3$ is a critical subdirect relation in Inv($w$), and

1. $w$ is a minimal WNU.
2. $\rho$ is compatible with $\sigma_3$.
3. (parallelogram property) $\forall a_1, a_2, a_3, b_1, b_2, b_3:
   (a_1, a_2, b_3), (b_1, b_2, a_3), (b_1, b_2, b_3) \in \rho \Rightarrow (a_1, a_2, a_3) \in \rho$.

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I need your help with Conjecture 1.

Let $\sigma_i$ be the minimal congruence on $A_i$ such that $w/\sigma_i$ is bijective.

**Conjecture 1**

Suppose $\rho \subseteq A_1 \times A_2 \times A_3$ is a critical subdirect relation in $\text{Inv}(w)$, and

1. $w$ is a minimal WNU.
2. $\rho$ is compatible with $\sigma_3$.
3. (parallelogram property) $\forall a_1, a_2, a_3, b_1, b_2, b_3:$
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Then $\rho$ is compatible with $\sigma_1$ and $\sigma_2$, or, equivalently,

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Thank you for your attention.
Algorithm

1-3 Preliminary steps. We repeat them if necessary.

4 If all domains are unary, we get a solution.

5 If there exists a binary absorption: Apply Absorbing Reduction, provide 1-consistency, and go to Step 4.

6 If there exists a center: Apply Central Reduction, provide 1-consistency, and go to Step 4.

7 If we get all functions after factorization: Apply “All Functions” reduction, provide 1-consistency, and go to Step 4.

8 If the WNU $\mathbf{w}$ is bijective after factorization

   1. solve the factorized CSP.
   2. if it has a solution, apply Linear Reduction and go to Step 4.
   3. if we can remove a constraint or split a variable to get a CSP instance $\Omega$ such that $\Omega^F$ has no solutions, we do this while possible.
   4. if we can remove a constraint or split a variable to get a CSP instance $\Omega$ such that $\Omega^F$ has a solution $(S_1, \ldots, S_n)$ but the reduction of $\Omega$ to $(S_1, \ldots, S_n)$ has no solutions, then we consider the reduction and go to Step 5.
   5. if $\Theta^F$ has no solutions and we cannot remove or split, then we reduce the original domain $A_i$ to $A'_i$. 

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