Group coextensions of monoids in an ordered setting

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Partially ordered monoids

Definition

A structure $(S; \cdot, \leqslant, 1)$ such that

(P1) $(S; \cdot, 1)$ is a commutative monoid,

(P2) $\leqslant$ is a compatible partial order
$(a \leqslant b$ implies $a \cdot c \leqslant b \cdot c)$

is called a (commutative) partially ordered monoid,
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Guiding example

A t-norm is a binary operation on the real unit interval, used in fuzzy logic to interpret the conjunction.

Given a t-norm $\odot \colon [0,1]^2 \to [0,1]$,

$$([0,1]; \odot, \leqslant, 1)$$

is a pomonoid.

Homomorphisms and coextensions

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A homomorphism $\pi\colon E \to S$ is a homomorphism of monoids that also preserves the order.

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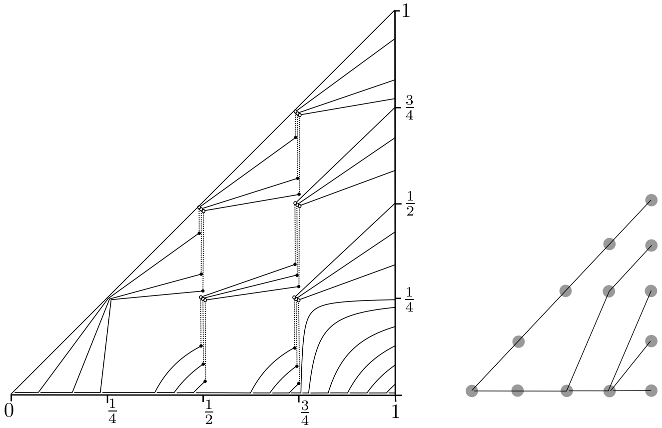
Let π be

- surjective
- and order-determining,
  i.e., for any $x, y \in E$, $\varphi(x) < \varphi(y)$ implies $x < y$.

In this case we call E a coextension of S.

Example



A t-norm-based pomonoid and its homomorphic image.

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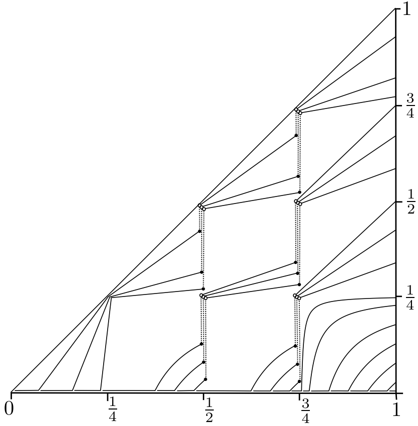

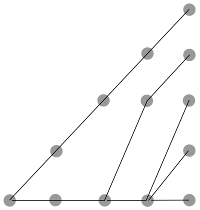

A t-norm-based pomonoid and its homomorphic image.

We are concerned with the opposite direction:
the coextensions of pomonoids.

Group coextensions à la Grillet/Leech

Let $\pi\colon E \to S$ be a surjective homomorphism of monoids such that the kernel of π is contained in $\mathcal{H}$.
Then E is a group coextension of S.

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Let $a \in S$ and $A = \{c \in E\colon \pi(c) = a\}$.

Put $T(A) = \{t \in E\colon t\,A \subseteq A\}$,
then the $\lambda_t^A\colon A \to A,\ c \mapsto t\,c$, where $t \in T(A)$,
form a group $\Gamma(A)$ – called Schützenberger group –,
which acts simply transitive on A.

Moreover, for $a, b \in S$ such that $b \leqslant_{\mathcal{H}} a$,
let $A = \pi^{-1}(a)$ and $B = \pi^{-1}(b)$.
Then $\varphi_B^A\colon \Gamma(A) \to \Gamma(B),\ \lambda_t^A \mapsto \lambda_t^B$ is a group homomorphism.

Preordered systems of groups

Definition

Let $(S; \preccurlyeq)$ be a preordered set.
For any $a \in S$, let $(G_a; +, 0)$ be a group,
and for any $a, b \in S$ such that $a \succcurlyeq b$,
let $\varphi^a_b \colon G_a \to G_b$ be a group homomorphism.

Assume that

- $\varphi^a_a = \mathrm{id}_{G_a}$ for any $a \in S$,
- $\varphi^b_c \circ \varphi^a_b = \varphi^a_c$ for any $a, b, c \in S$ such that $a \succcurlyeq b \succcurlyeq c$.

Then $G = (G_a)_{a \in S}$ and $\varphi = (\varphi^a_b)_{a,b \in S,\ a \succcurlyeq b}$ is called a preordered system of groups over $(S; \preccurlyeq)$.

Group coextensions of monoids

Theorem (P.A. Grillet; J. Leech)

Let $(S;\cdot,1)$ be a monoid. Let (G,φ) be a preordered system of groups over $(S;\leqslant_{\mathcal{H}})$. For each $a,b\in S$, let $\sigma_{a,b}\in G_{ab}$ be such that:

1. $\sigma_{1,1} = 0$;
2. $\sigma_{a,b} = \sigma_{b,a}$ for any $a,b\in S$;
3. $\varphi^{ab}_{abc}(\sigma_{a,b})+\sigma_{ab,c} = \varphi^{bc}_{abc}(\sigma_{b,c})+\sigma_{a,bc}$ for any $a,b,c\in S$.

Let then

$$E = \{(a,x)\colon a\in S,\ x\in G_a\},$$

endowed with the product

$$(a,x)\,(b,y) = (a\,b,\ \varphi^{a}_{ab}(x)+\varphi^{b}_{ab}(y)+\sigma_{a,b}).$$

Then $(E;\cdot,(1,0))$ is a group coextension of S.

Adaptation and generalisation

We shall

- **take into account a partial order;**
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Definition

Let $(S; \preccurlyeq)$ be a preordered set.
For any $a \in S$, let $(M_a; +, \leqslant_a, 0)$ be a pomonoid, and for $a, b \in S$ such that $a \succcurlyeq b$, let $\varphi^a_b \colon M_a \to M_b$ be a (pomonoid) homomorphism.

Assume that

- $\varphi^a_a = \mathrm{id}_{M_a}$ for any $a \in S$,
- $\varphi^b_c \circ \varphi^a_b = \varphi^a_c$ for any $a, b, c \in S$ such that $a \succcurlyeq b \succcurlyeq c$.

Then $M = (M_a)_{a \in S}$ and $\varphi = (\varphi^a_b)_{a,b \in S,\ a \succcurlyeq b}$ is called a preordered system of pomonoids over $(S; \preccurlyeq)$.

Group coextensions of monoids

Theorem (J. Janda, Th. V.)

Let $(S;\cdot,1)$ be a pomonoid. Let (M,φ) be a preordered system of pomonoids over $(S;\leqslant_{\mathcal{H}})$. For each $a,b\in S$, let $\sigma_{a,b}\in M_{ab}$ be such that:

1. $\sigma_{1,a} \;=\; 0$ for any $a\in S$;
2. $\sigma_{a,b} \;=\; \sigma_{b,a}$ for any $a,b\in S$;
3. $\varphi^{ab}_{abc}(\sigma_{a,b})+\sigma_{ab,c} \;=\; \varphi^{bc}_{abc}(\sigma_{b,c})+\sigma_{a,bc}$ for any $a,b,c\in S$.
4. if, for $a,b,c\in S$, $a<b$ and $a\,c=b\,c$, then $\varphi^{a}_{ac}(x)+\sigma_{a,c}\leqslant_{ac}\varphi^{b}_{bc}(y)+\sigma_{b,c}$ for any $x\in M_a$ and $y\in M_b$.

Let then $E \;=\; \{(a,x)\colon a\in S,\ x\in M_a\}$ be endowed with

$$
\begin{aligned}
(a,x)\cdot(b,y) &\;=\; (a\,b,\ \varphi^{a}_{ab}(x)+\varphi^{b}_{ab}(y)+\sigma_{a,b}),\\
(a,x)\leqslant_E(b,y) &\;\text{ if } a<b, \text{ or } a=b \text{ and } x\leqslant_a y.
\end{aligned}
$$

Then $(E;\cdot,(1,0))$ is a coextension of S.

Example

Let $S = \{-3, -2, -1, 0\}$ be the four-element chain and put

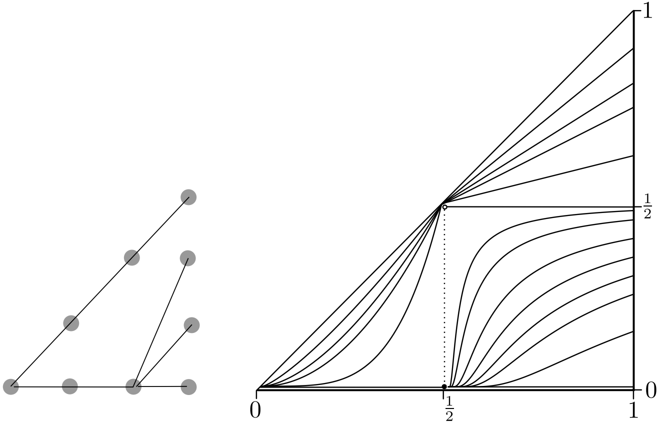
$$a \cdot b \; = \; \begin{cases} a & \text{if } b = 0, \\ b & \text{if } a = 0, \\ -3 & \text{otherwise.} \end{cases}$$

We define a preordered system (M, φ) of pomonoids over $(S, \leqslant_{\mathcal{H}})$:

$$M_0 \; = \; \mathbb{R}^-, \quad M_{-2} = \mathbb{R}, \quad M_{-1} = M_{-3} = \{0\};$$
$$\varphi^0_{-2} \colon M_0 \to M_{-2}, \; x \mapsto x.$$

In addition, we let $\sigma_{a,b} = 0$ in all cases.

Example



A four-element pomonoid and its coextension to a t-norm-based pomonoid.

Conclusion

- The theory of group coextensions of monoids can be modified to the case that (i) a compatible order is present and (ii) pomonoids are used as extending structures.

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Conclusion

- The theory of group coextensions of monoids can be modified to the case that (i) a compatible order is present and (ii) pomonoids are used as extending structures.
- The method covers an amazingly large amount of those coextensions of pomonoids that arise in the context of t-norms used in fuzzy logic.
- The method has its limits; certainly not all coextensions are due to homomorphisms between the congruence classes seen as pomonoids.

Outlook: towards a more general framework

Let $(E; \wedge, \vee, \cdot, \to, 1)$ be a residuated ℓ-monoid, that is, $(E; \wedge, \vee, \cdot, 1)$ is a lattice-ordered monoid and

$$a \cdot b \leqslant c \quad \text{iff} \quad a \leqslant b \to c.$$

In this case, each congruence ϑ is determined by the class $H = [1]_\vartheta$, which is a convex subalgebra.

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Coextension problem

To which extent is E determined by the quotient E/H, the algebra H, and the lattice order of E?

Outlook: towards a more general framework

We observe:

- Each congruence class C is an H-poset:

$$\lambda_h c = h \cdot c, \quad \text{where } h \in H,\ c \in C.$$

- The multiplication restricted to a pair C and D of congruence classes is "bilinear" w.r.t. to the action of H:

$$\lambda_h c \cdot d = c \cdot \lambda_h d = \lambda_h (c \cdot d), \quad \text{where } h \in H,\ c, d \in C.$$

  Hence it can be identified with a homomorphism from the tensor product of the H-posets C and D to $C \cdot D$.

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Hence we need for the coextension:

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E.g., the case that E is a chain and $H = \mathbb{R}^-$ seems tractable.