LATTICES WITHOUT ABSORPTION

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BISEMILATTICES

A **bisemilattice** is an algebra $(B, \cdot, +)$ with two semilattice operations $\cdot$ and $+$, the first interpreted as a meet and the second as a join.

A **Birkhoff system** is a bisemilattice satisfying a weakened version of the absorption law for lattices known as *Birkhoff's equation*:

$$x \cdot (x + y) = x + (x \cdot y).$$

Each bisemilattice induces two partial orderings on its underlying set:

$$x \leq_{\cdot} y \text{ iff } x \cdot y = x,$$
$$x \leq_{+} y \text{ iff } x + y = y.$$

EXAMPLES

Lattices: $x + xy = x(x + y) = x$,
and $\leq_{\cdot} = \leq_{+}$.

(Stammered) semilattices: $x \cdot y = x + y$,
and $\leq_{\cdot} = \geq_{+}$.

Bichains: both meet and join reducts are chains, e.g. 2-element lattice $\mathbf{2}_l$,
2-element semilattice $\mathbf{2}_s$,
and the four non-lattice and non-semilattice 3-element bichains:

3	1	3	2	3	3	3	2
2	3	2	1	2	1	2	3
1	2	1	3	1	2	1	1
$\mathbf{3}_d$		$\mathbf{3}_n$		$\mathbf{3}_j$		$\mathbf{3}_m$	

EXAMPLES, cont.

Meet-distributive Birkhoff systems:
$x(y + z) = xy + xz$ (MD),
e.g. $\mathbf{3}_m$.

Join-distributive Birkhoff systems:
$x + yz = (x + y)(x + z)$ (JD),
e.g. $\mathbf{3}_j$.

Distributive Birkhoff systems:
satisfy both (MD) and (JD),
e.g. $\mathbf{3}_d$.

Quasilattices:
$(x + y)z + yz = (x + y)z$ (mQ),
$(xy + z)(y + z) = xy + z$ (jQ),
or equivalently:
$x + y = x \Rightarrow (xz) + (yz) = xz$,
$xy = x \Rightarrow (x + z)(y + z) = x + z$.

SEMILATTICE SUMS

Each Birkhoff system A has a homomorphism onto a semilattice.

The greatest semilattice homomorphic image $S = h(A)$ of A is called the **semilattice replica** of A.
Its kernel ker$h$ is called the **semilattice (replica) congruence** of A.

If the blocks of ker$h$ are all lattices A_s, with $s \in S$, then A is said to be **the semilattice sum of lattices** A_s and is denoted $\bigsqcup_{s \in S} A_s$.

PŁONKA SUMS

The semilattice sum $\bigsqcup_{s\in S} A_s$ of lattices A_s is **functorial**, if there is a functor

$$F : S \to \mathsf{L}; (s \to t) \mapsto (\varphi_{s,t} : A_s \to A_t)$$

from the category S to the category L of lattices, assigning to each morphism $s \to t$ of S a homomorphism $\varphi_{s,t} : A_s \to A_t$ of lattices.

The functorial sum $\bigsqcup_{s\in S} A_s$ becomes the **Płonka sum** (of lattices A_s over the semilattice S by the functor F),
by defining, for $a_s \in A_s, b_t \in A_t$, their join and meet as follows:

$$a_s + b_t = a_s\varphi_{s,s+t} + b_t\varphi_{t,s+t},$$

$$a_s \cdot b_t = a_s\varphi_{s,s+t} \cdot b_t\varphi_{t,s+t}.$$

The Płonka sum of Birkhoff systems is a Birkhoff system.

REGULARIZATION and ...

An equation $p = q$ is **regular** if the same variables appear on each side.
A variety is **regular** if all equations valid in it are regular.

Proposition A variety of Birkhoff systems is irregular precisely, if it is a variety of lattices. The variety of semilattices is the smallest regular variety.

For each irregular variety V of Birkhoff systems, there is a smallest regular variety $\widetilde{\mathrm{V}}$ containing V, called the **regularization** of V. It is defined by all regular equations that are valid in V.

The regularization $\widetilde{\mathrm{V}}$ consists precisely of Płonka sums of bisemilattices in V.

...quasilattices

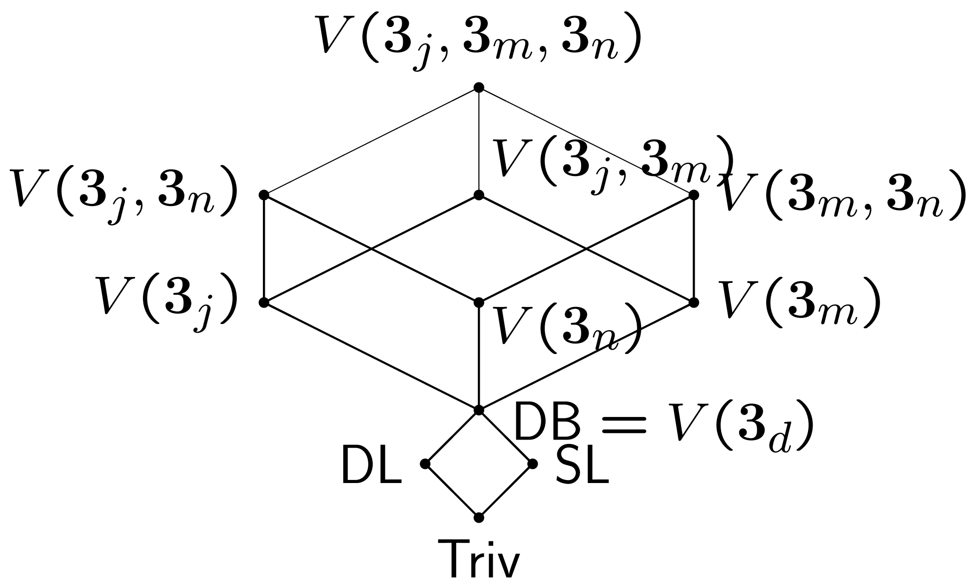
Theorem (Padmanabhan) Each variety of quasilattices is the regularization of a variety of lattices, and consists precisely of Płonka sums of lattices in this variety.

Corollary (Płonka) The regularization $\widetilde{\mathrm{DL}}$ of the variety DL of distributive lattices consists of Płonka sums of distributive lattices, and is generated by the distributive 3-element bichain $\mathbf{3}_d$.

Theorem(Dudek, Graczyńska) For a variety V of lattices, the lattice $\mathcal{L}(\widetilde{\mathrm{V}})$ of subvarieties of its regularization $\widetilde{\mathrm{V}}$ is isomorphic to the direct product $\mathcal{L}(\mathrm{V}) \times 2$ of the lattice of subvarieties of V and the 2-element lattice 2.

VARIETIES GENERATED BY 3-ELEMENT BICHAINS

For Birkhoff systems $A_1, \ldots, A_n$, let $V(A_1, \ldots, A_n)$ denote the variety of Birkhoff systems generated by $A_1, \ldots, A_n$.



SPLITTINGS

A pair (u, w) of elements of a complete lattice L is called a **splitting pair** or briefly a **splitting** of L, if L is the disjoint union $(u] \cup [w)$ of the set of elements that are underneath of u and the set of elements that are above of w.

Proposition(McKenzie, Jipsen-Rose) Let (U, W) be a splitting pair of subvarieties of V. Then there is a subdirectly irreducible algebra S in V that generates W. The variety U is the largest subvariety of V that does not contain S. It is defined by the equations satisfied in V and one additional equation.

The subdirectly irreducible algebra S is called a **splitting algebra** in V, the variety U is called the **splitting variety** of S, and the additional equation defining the splitting variety of S is called the **splitting equation** for S.

An algebra P in a variety V is **weakly projective** in V if for any algebra $A \in$ V and any homomorphism $f : A \to P$ onto P there is a subalgebra B of A such that the restriction $f|_B : B \to P$ is an isomorphism.

For a variety V and an algebra S in V, define

$$\mathsf{V}_S = \{A \in \mathsf{V} \mid S \not\leq A\}.$$

Proposition(Jipsen-Rose) Let S be an algebra that is subdirectly irreducible and weakly projective in a variety V. Then S is a splitting algebra in V and $(\mathsf{V}_S, V(S))$ is a splitting pair of subvarieties of V.

Theorem (Harding, C. Walker, E. Walker) A finite bichain is weakly projective in the variety BS if, and only if, it does not contain a subalgebra isomorphic to $\mathbf{3}_d$.

EXAMPLES OF SPILTTINGS

Proposition The splitting variety $\mathsf{BS}_{\mathbf{2}_l}$ of $\mathbf{2}_l$ is the variety SL of semilattices, and the splitting equation *($S_{\mathbf{2}_l}$)* is $xy = x + y$.

Proposition The splitting variety $\mathsf{BS}_{\mathbf{2}_s}$ of $\mathbf{2}_s$ is the variety L of lattices, and its splitting equation *($S_{\mathbf{2}_s}$)* is absorption, $x + xy = x$.

Proposition Each of the bichains $\mathbf{3}_m$, $\mathbf{3}_j$ and $\mathbf{3}_n$ is subdirectly irreducible and weakly projective. Their splitting equations are the following.

$$(z + xyz)(z + yz + xyz) = z + xyz, \qquad (\mathsf{S}_{\mathbf{3}_m})$$

$$z(x+y+z)+z(y+z)(x+y+z) = z(x+y+z), \qquad (\mathsf{S}_{\mathbf{3}_j})$$

$$(z + xyz)(z + yz + xyz) = z + yz + xyz. \qquad (\mathsf{S}_{\mathbf{3}_n})$$

These equations define the varieties $\mathsf{BS}_{\mathbf{3}_m}$, $\mathsf{BS}_{\mathbf{3}_j}$ and $\mathsf{BS}_{\mathbf{3}_n}$, respectively.

A STRUCTURE THEOREM

We give a structure theorem for the variety $V(\mathsf{S}_{\mathbf{3}_m}, \mathsf{S}_{\mathbf{3}_j})$ defined by $\mathsf{S}_{\mathbf{3}_m}$ and $\mathsf{S}_{\mathbf{3}_j}$, and in particular for its subvariety $V(\mathbf{3}_n)$.

The variety $V(\mathsf{S}_{\mathbf{3}_m}, \mathsf{S}_{\mathbf{3}_j})$ is defined by the splitting equations of the bichains $\mathbf{3}_m$ and $\mathbf{3}_j$. Thus a Birkhoff system belongs to $V(\mathsf{S}_{\mathbf{3}_m}, \mathsf{S}_{\mathbf{3}_j})$ if, and only if, it contains no subalgebra isomorphic to either $\mathbf{3}_m$ or $\mathbf{3_j}$.

Let A be a Birkhoff system. We say that a subset $S \subseteq A$ is a **sublattice** of A if S is a subalgebra of A that is a lattice. We say that S is a **convex sublattice** of A if S is a sublattice of A and is convex in each semilattice reduct of A.

For a Birkhoff system A, define a binary relation θ on A by setting $a\,\theta\,b$ if a and b generate a sublattice of A.

Theorem If $A \in V(\mathsf{S}_{\mathbf{3}_m}, \mathsf{S}_{\mathbf{3}_j})$, then θ is a bisemilattice congruence of A, the equivalence classes of θ are convex sublattices, and the quotient A/θ is a semilattice.

In particular, the Birkhoff system A is a semilattice sum of lattices $A_s = a/\theta$ over the semilattice $S = A/\theta$.

Proposition In a semilattice sum $\coprod_{s \in S} A_s$, the summands A_s are necessarily convex sublattices of A, and the congruence θ is unique.

Corollary A Birkhoff system A belongs to the variety $V(\mathsf{S}_{\mathbf{3}_m}, \mathsf{S}_{\mathbf{3}_j})$ if, and only if, it is a semilattice sum of lattices.

Corollary Each member of the variety $V(\mathbf{3}_n)$ is a semilattice sum of distributive lattices.

MAL'CEV PRODUCT

Let V and W be two varieties of Birkhoff systems. Then the **Mal'cev product** V $\circ$ W of V and W consists of Birkhoff systems A with a congruence φ such that the quotient A/φ is in W, and each congruence class a/φ of A is in V.

Corollary The class of Birkhoff systems that are semilattice sums of lattices is the Mal'cev product L $\circ$ SL of the varieties L of lattices and SL of semilattices within the class of Birkhoff systems.

Corollary The following three classes of Birkhoff systems are equal:
(a) the variety $V(\mathsf{S}_{\mathbf{3}_m}, \mathsf{S}_{\mathbf{3}_j})$,
(b) the class of Birkhoff systems that are semilattice sums of lattices,
(c) the quasivariety L $\circ$ SL.

Reconstruction

There is a general method of reconstructing a semilattice sum of lattices from the summands and the quotient, by means of so-called strict Lallement sums.
In the case of sums of bounded lattices, such sums have a more direct description.

Let $(S, +, \cdot)$, where $x + y = x \cdot y$, be a semilattice, and let A_s, for $s \in S$, be bounded lattices, where 0_s and 1_s are the bounds of A_s.
For $s \cdot t = s + t = t$ in S, let the map

$$\varphi_{s,t} : (A_s, \cdot) \to (A_t, \cdot)$$

be a homomorphism of the meet-semilattice reduct, and the map

$$\psi_{s,t} : (A_s, +) \to (A_t, +)$$

be a homomorphism of the join semilattice reduct of A. Let $\varphi_{s,s}$ and $\psi_{s,s}$ be identity maps.

The **strict Lallement sum** of the lattices A_s over the semilattice S by the mappings $\varphi_{s,t}$ and $\psi_{s,t}$ is the disjoint union of the A_s (with $s \in S$) with operations $\cdot$ and $+$ defined for $a_s \in A_s$ and $a_t \in A_t$ by

$$a_s + b_t = a_s\varphi_{s,s+t} + b_t\varphi_{t,s+t},$$

$$a_s \cdot b_t = a_s\varphi_{s,s\cdot t} \cdot b_t\varphi_{t,s\cdot t}.$$

Theorem Let A be a Birkhoff system. Then A is a semilattice sum $\bigsqcup_{s\in S} A_s$ of bounded lattices A_s over a semilattice S if, and only if, it is a strict Lallement sum of the lattices A_s over the semilattice S given by the homomorphisms $\varphi_{s,t}$ and $\psi_{s,t}$ described above.

Corollary Each finite algebra in the variety $V(\mathsf{S}_{\mathbf{3}_m}, \mathsf{S}_{\mathbf{3}_j})$ is a strict Lallement sum of lattices, and each finite algebra in the variety $V(\mathbf{3}_n)$ is a strict Lallement sum of distributive lattices.

Problem Is each semilattice sum of lattices embeddable into a semilattice sum of bounded lattices?