Taylor's modularity conjecture for idempotent varieties

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Congruence modular varieties

A variety is congruence modular if every algebra in it and every a triple of congruences α, β, and γ such that $\alpha \geq \gamma$ we have

$$\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee \gamma.$$

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Theorem (A. Day, 1969)

A variety $\mathcal{V}$ is congruence modular if and only if there are terms $d_0, \ldots, d_n$ such that

$$\begin{aligned} d_0(x, y, z, w) &\approx x, \quad d_n(x, y, z, w) \approx w, \\ d_i(x, y, y, x) &\approx x, \text{ for all } i, \\ d_i(x, x, y, y) &\approx d_{i+1}(x, x, y, y), \text{ for all even } i, \text{ and} \\ d_i(x, y, y, z) &\approx d_{i+1}(x, y, y, z), \text{ for all odd } i. \end{aligned}$$

Taylor's modularity conjecture

Conjecture (Taylor)

Suppose that Σ_1 and Σ_2 are two sets of identities in disjoint languages such that neither of them implies existence of Day terms. Then $\Sigma_1 \cup \Sigma_2$ does not imply existence of Day terms either.

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O.C. Garcia and Walter Taylor, *The lattice of interpretability types of varieties*. Mem. Amer. Math. Soc., 50:v+125, 1984.

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(For Σ_1 take Mal'cev term identities, and for Σ_2 Jónsson chain of length ≥ 2.)

The lattice of interpretability types of varieties

An interpretation of a variety $\mathcal{V}$ in a variety $\mathcal{W}$ is a mapping I of basic operations of $\mathcal{V}$ to terms of $\mathcal{W}$ such that

$$\mathcal{V} \models t \approx s \to \mathcal{W} \models I(t) \approx I(s).$$

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Interpretability is a partial order of the class of all varieties, and after factoring out the equi-interpretable varieties, we get a (class-size) latticed ordered poset.

Reformulation of the conjecture

An interpretability join of two varieties $\mathcal{V}$ and $\mathcal{W}$ is the variety whose signature is the disjoint union of signatures of $\mathcal{V}$ and $\mathcal{W}$ axiomatized by the union of $\mathrm{Eq}(\mathcal{V})$ and $\mathrm{Eq}(\mathcal{W})$.

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Conjecture (Alternative formulation)

For any two varieties $\mathcal{V}$ and $\mathcal{W}$ which are not congruence modular, their interpretability join is not congruence modular either.

Stop this abstract nonsense!

Theorem (Kearnes, Kiss, 2013)

An idempotent variety satisfies a non-trivial congruence identity if and only if it is not interpretable in the variety of semilattices.

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Looking for the right relational structure

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A *pentagon* is a structure $\mathbb{P}$ over the signature $\{\alpha, \beta, \gamma\}$, all binary relations which are equivalence relations on P that satisfy

- $\alpha \leq \beta$,
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Lemma (Bova, Chen, and Valeriote)

If a locally finite variety is not congruence modular, then there is a finite algebra $\mathbf{A}$ *in the variety with three congruences* α, β, *and* γ *such that* $(A, \alpha, \beta, \gamma)$ *is a disjoint union of pentagons of which at least one is interesting.*

Looking for the right relational structure

Notation

For a congruence α of a product $\mathbf{A} \times \mathbf{B}$ and $a \in A$ let α^a denotes the equivalence

$$\{(b, b') : ((a, b), (a, b')) \in \alpha\}$$

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Lemma (McGarry, 2009)

A locally finite idempotent variety is not congruence modular if and only if it contains algebras $\mathbf{A}$ *and* $\mathbf{B}$ *with a congruence* $\alpha \leq \operatorname{Ker} \pi_{\mathbf{A}}$ *of* $\mathbf{A} \times \mathbf{B}$ *such that*

- $\alpha^a = 1_B$ *for some* $a \in A$*,*
- *if* $\alpha^a \neq 1_B$ *then* $\alpha_a = \eta$ *for some fixed* $\eta < 1_B$*.*

Modularity blocker

We say that a congruence $\alpha \leq \operatorname{Ker} \pi_{\mathbf{A}}$ of $\mathbf{A} \times \mathbf{B}$ is a modularity blocker if there exists $\eta \in \operatorname{Con} \mathbf{B}$ such that

- $\alpha^a = 1_B$ for at least one $a \in A$, and
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Theorem (O, 2016)

An idempotent variety is not congruence modular if and only if $\mathbf{F}(x, y) \times \mathbf{F}(x, y)$ *has a modularity blocker.*

A corollary

Definition

A pentagon $(P, \alpha, \beta, \gamma)$ is *special* if

- $P = A \times B$,
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Corollary

An idempotent variety that is not congruence modular is interpretable in the idempotent reduct of a special interesting pentagon.

Back to Taylor's conjecture

For an infinite cardinal κ fix $U_\kappa \subseteq \kappa$ with $|U_\kappa| = |\kappa \setminus U_\kappa| = \kappa$, let $\mathbb{P}_\kappa$ denotes a special pentagon $(P_\kappa, \alpha, \beta, \gamma)$ with $P_\kappa = \kappa \times \kappa$ and $\alpha^a = 1_\kappa$ if $a \in U_\kappa$, and $\alpha^a = 0_\kappa$, otherwise.

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Lemma

For every special interesting pentagon P there exists a clone homomorphism from its polymorphism clone to the polymorphism clone of $\mathbb{P}_\kappa$ for all infinite cardinals $\kappa \geq |P|$.

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Lemma

For every special interesting pentagon P there exists a clone homomorphism from its polymorphism clone to the polymorphism clone of $\mathbb{P}_\kappa$ for all infinite cardinals $\kappa \geq |P|$.

Theorem (O., 2016)

Every idempotent variety that is not congruence modular is interpretable in the variety generated by $(P_\kappa, \mathrm{Pol}\,\mathbb{P}_\kappa)$ for all large enough κ.

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Thank you for your attention!

Identities satisfied by $\operatorname{Pol}\mathbb{P}_\kappa$ but not by $\operatorname{Pol}\mathbb{P}_\lambda$ for $\lambda < \kappa$

Functions f_i, $i \in \kappa$ are binary and $p_{i,j}$, and $q_{i,j}$, $r_{i,j}$, $i, j \in \kappa$ are ternary.

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$$\begin{aligned} x &\approx p_{i,j}(x, f_j(x,y), y), \\ p_{i,j}(x, f_i(x,y), y) &\approx q_{i,j}(x, f_j(x,y), y), \\ q_{i,j}(x, f_i(x,y), y) &\approx r_{i,j}(x, f_j(x,y), y), \\ r_{i,j}(x, f_i(x,y), y) &\approx y \end{aligned}$$

for all $i \neq j$, and $f_i(x,x) \approx x$ for all i.

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for all $i \neq j$, and $f_i(x,x) \approx x$ for all i.

Corollary

The set of all interpretability classes of idempotent varieties that are not congruence modular does not have a largest element.