Permutation classes closed under pattern involvement and composition

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Group classes

M. D. ATKINSON, R. BEALS, Permuting mechanisms and closed classes of permutations, in: C. S. Calude, M. J. Dinneen (eds.), *Combinatorics, Computation & Logic,* Proc. DMTCS '99 and CATS '99 (Auckland), Aust. Comput. Sci. Commun., 21, No. 3, Springer, Singapore, 1999, pp. 117–127.

M. D. ATKINSON, R. BEALS, Permutation involvement and groups, *Q. J. Math.* **52** (2001) 415–421.

Group classes

Theorem (Atkinson, Beals)

If C is a permutation class in which every level $C^{(n)}$ is a permutation group, then the level sequence $C^{(1)}, C^{(2)}, \dots$ eventually coincides with one of the following families of groups:

(1) *the groups $S_n^{a,b}$ for some fixed $a, b \in \mathbb{N}_+$,*

(2) *the natural cyclic groups Z_n,*

(3) *the full symmetric groups S_n,*

(4) *the groups $\langle G_n, \delta_n \rangle$, where $(G_n)_{n \in \mathbb{N}}$ is one of the above families (with $a = b$ in (1)).*

$\delta_n = n(n-1)\dots 21$	descending permutation
$\zeta_n = (1\ 2\ \cdots\ n) = 23\dots n1$	natural cycle
$Z_n = \langle \zeta_n \rangle$	natural cyclic group
$D_n = \langle \zeta_n, \delta_n \rangle$	natural dihedral group

Group classes

Theorem (Atkinson, Beals)

Let C be a permutation class in which every level $C^{(n)}$ is a transitive group. Then, with the exception of at most two levels, one of the following holds.

(1) $C^{(n)} = S_n$ *for all* $n \in \mathbb{N}_+$.

(2) *For some* $M \in \mathbb{N}$, $C^{(n)} = S_n$ *for* $1 \leq n \leq M$, *and* $C^{(n)} = D_n$ *for* $n > M$.

(3) *For some* $M, N \in \mathbb{N}$ *with* $M \leq N$, $C^{(n)} = S_n$ *for* $1 \leq n \leq M$, $C^{(n)} = D_n$ *for* $M + 1 \leq n \leq N$, *and* $C^{(n)} = Z_n$ *for* $n > N$.

The exceptions, if any, may occur in the second and third cases and are of the following two possible types:

(i) $C^{(M+1)} = A_{M+1}$ *and* $C^{(M+2)}$ *is an anomalous group that is neither* D_{M+2} *nor* Z_{M+2}, *or*

(ii) $C^{(M+1)}$ *is a proper overgroup of* Z_{M+1} *but is not* D_{M+1}.

Goal

We would like to describe the sequence

$$G, \operatorname{Comp}^{(n+1)} G, \operatorname{Comp}^{(n+2)} G, \ldots$$

for an arbitrary group $G \leq S_n$.

We would also like to determine how fast this sequence reaches the asymptotic behaviour predicted by Atkinson and Beals's results.

Literature

E. Lehtonen, R. Pöschel, Permutation groups, pattern involvement, and Galois connections, arXiv:1605.04516.

E. Lehtonen, Permutation groups arising from pattern involvement, arXiv:1605.05571.

Roadmap

- S_n, $\langle \delta_n \rangle$, trivial
- A_n
- $\zeta_n \in G$ and $A_n \nleq G$
- $\zeta_n \notin G$:
  - intransitive
  - transitive:
    - imprimitive
    - primitive

$\iota_n = 12 \ldots n$	ascending (identity) permutation
$\delta_n = n(n-1) \ldots 21$	descending permutation
$\zeta_n = (1\ 2\ \cdots\ n) = 23 \ldots n1$	natural cycle
$Z_n = \langle \zeta_n \rangle$	natural cyclic group
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Simple observations

Lemma

Let $n, m \in \mathbb{N}_+$ with $n \leq m$. Let $G \leq S_n$. Then $\delta_m \in \mathrm{Comp}^{(m)} G$ if and only if $\delta_n \in G$.

Lemma

Let $G \leq S_n$.

(a) *The following statements are equivalent.*
  (i) $Z_n \leq G$.
  (ii) $Z_{n+1} \leq \mathrm{Comp}^{(n+1)} G$.
  (iii) $\mathrm{Comp}^{(n+1)} G$ *contains a permutation* $\pi \in Z_{n+1} \setminus \{\iota_{n+1}\}$.

(b) *The following statements are equivalent.*
  (i) $D_n \leq G$.
  (ii) $D_{n+1} \leq \mathrm{Comp}^{(n+1)} G$.
  (iii) $\mathrm{Comp}^{(n+1)} G$ *contains a permutation* $\pi \in D_{n+1} \setminus (Z_{n+1} \cup \{\delta_{n+1}\})$.

Symmetric, trivial, . . .

Theorem

The following statements hold for all $n \in \mathbb{N}_+$.

(a) $\mathrm{Comp}^{(n+1)} S_n = S_{n+1}$.

(b) *If $n \geq 2$, then* $\mathrm{Comp}^{(n+1)} \{\iota_n\} = \{\iota_{n+1}\}$.

(c) *If $n \geq 3$, then* $\mathrm{Comp}^{(n+1)} \langle\delta_n\rangle = \langle\delta_{n+1}\rangle$.

Notation

Let Π be a partition of $[n]$.

$$S_\Pi := \{\pi \in S_n \mid \forall B \in \Pi \colon \pi(B) = B\}$$

Alternating groups

$Œ_{n+1}$ – partition of $[n+1]$ into odd and even numbers
$S_{Œ_{n+1}}$ – permutations preserving blocks of $Œ_{n+1}$
$W_{Œ_{n+1}}$ – permutations interchanging blocks of $Œ_{n+1}$
A_{n+1} – even permutations
O_{n+1} – odd permutations

Theorem

$$\operatorname{Comp}^{(n+1)} A_n = (S_{Œ_{n+1}} \cap A_{n+1}) \cup (W_{Œ_{n+1}} \cap O_{n+1}).$$

Theorem

$$\operatorname{Comp}^{(n+2)} A_n = \begin{cases} \langle \delta_{n+2} \rangle, & \text{if } n \equiv 0 \pmod 4, \\ Z_{n+2}, & \text{if } n \equiv 1 \pmod 4, \\ \{\iota_{n+2}\}, & \text{if } n \equiv 2 \pmod 4, \\ D_{n+2}, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Groups containing the natural cycle

Theorem

Let $G \leq S_n$, and assume that G contains the natural cycle ζ_n.

(i) *If $D_n \leq G$ and $G \notin \{S_n, A_n\}$, then $\operatorname{Comp}^{(n+1)} G = D_{n+1}$.*

(ii) *If $D_n \nleq G$, then $\operatorname{Comp}^{(n+1)} G = Z_{n+1}$.*

Intransitive groups

Let $G \leq S_n$ be an intransitive group.

Then $G \leq S_{\operatorname{Orb} G}$, where $\operatorname{Orb} G$ be the set of orbits of G.

Moreover, $\operatorname{Orb} G$ is the finest partition Π such that $G \leq S_\Pi$.

Intransitive groups

Let Π be a partition of $[n]$.

$\Pi = \{\{1, 2, 3, 7, 8, 9, 10\}, \{4, 5, 6, 12, 13, 14\}, \{11\}\}$

Define the partition Π' of $[n + 1]$ as follows.

Let I_Π be the coarsest interval partition that refines Π.

$I_\Pi = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9, 10\}, \{11\}, \{12, 13, 14\}\}$

For each $[a, b] \in I_\Pi$, we let $\{a\}$ and $[a + 1, b]$ be blocks of Π'.

Exceptions:

If $a = 1$ and $b \neq n$, then $[a, b]$ is a block of Π'.

If $a \neq 1$ and $b = n$, then $\{a\}$ and $[a + 1, n + 1]$ are blocks of Π'.

If $a = 1$ and $b = n$, then $[1, n + 1]$ is a block of Π'.

$\Pi' = \{\{1, 2, 3\}, \{4\}, \{5, 6\}, \{7\}, \{8, 9, 10\}, \{11\}, \{12\}, \{13, 14, 15\}\}$

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Intransitive groups

Theorem

Let Π be a partition of $[n]$. Then, for all $i \geq 1$, we have

$$\operatorname{Comp}^{(n+i)} S_\Pi = \begin{cases} S_{\Pi^{(i)}}, & \text{if } \delta_n \notin S_\Pi, \\ \langle S_{\Pi^{(i)}}, \delta_{n+1} \rangle, & \text{if } \delta_n \in S_\Pi. \end{cases}$$

$\Pi^{(1)} := \Pi'$
$\Pi^{(i+1)} := (\Pi^{(i)})' \quad (i \geq 1)$

Intransitive groups

Theorem

Let $G \leq S_n$ be an intransitive group, and let $\Pi := \operatorname{Orb} G$. Let a and b be the largest numbers α and β, respectively, such that $S_n^{\alpha,\beta} \leq G$. Then for all $\ell \geq M_{a,b}(\Pi)$, it holds that $\operatorname{Comp}^{(n+\ell)} G = S_{n+\ell}^{a,b}$ or $\operatorname{Comp}^{(n+\ell)} G = \langle S_{n+\ell}^{a,b}, \delta_{n+\ell} \rangle$.

$$M(\Pi) := \max(\{|B| : B \in I_\Pi^-\} \cup \{1\})$$
$$M_{a,b}(\Pi) := \max(M(\Pi), |1/I_\Pi| - a + 1, |n/I_\Pi| - b + 1)$$

Notation

Let Π be a partition of $[n]$.

$$\operatorname{Aut} \Pi := \{\pi \in S_n \mid \forall B \in \Pi \colon \pi(B) \in \Pi\}$$

Imprimitive groups

Theorem

Let Π *be a partition of* $[n]$ *with no trivial blocks. Then*

$$\operatorname{Comp}^{(n+1)} \operatorname{Aut} \Pi = \begin{cases} \langle S_{\Pi'}, E_\Pi \rangle, & \text{if } \delta_n \notin \operatorname{Aut} \Pi, \\ \langle S_{\Pi'}, E_\Pi, \delta_{n+1} \rangle, & \text{if } \delta_n \in \operatorname{Aut} \Pi, \end{cases}$$

where E_Π *is the following set of permutations:*

- *If* $[1, \ell] \propto \Pi$ *for some* ℓ *with* $1 < \ell < n$*, then* $\nu_\ell^{(n+1)} \in E_\Pi$*.*
- *If* $[m, n] \propto \Pi$ *for some* m *with* $1 < m < n$*, then* $\lambda_{n-m+1}^{(n+1)} \in E_\Pi$*.*
- *If* $[1, n] \propto \Pi$*, then* $\zeta_{n+1} \in E_\Pi$*.*
- E_Π *does not contain any other elements.*

Primitive groups

Theorem

Assume that $G \leq S_n$ is a primitive group such that $\zeta_n \notin G$ and $A_n \nleq G$.

(i) (a) $n = 6$

G	$\mathrm{Comp}^{(n+1)}\, G$
$\langle (1\ 2\ 3\ 4), (3\ 4\ 5\ 6) \rangle$	$\{1234567, 2154376, 6734512, 7654321\}$
$\langle (1\ 2\ 3\ 4), (2\ 3\ 4\ 5\ 6) \rangle$	$\{1234567, 1276543, 1543276, 1567234\}$
$\langle (1\ 2\ 3\ 4\ 5), (3\ 4\ 5\ 6) \rangle$	$\{1234567, 2165437, 4561237, 5432167\}$
$\langle (1\ 2\ 3\ 4\ 5), (1\ 3\ 4)(2\ 5\ 6) \rangle$	$\langle \nu_5^{(7)} \rangle$
$\langle (2\ 3\ 4\ 5\ 6), (1\ 2\ 5)(3\ 4\ 6) \rangle$	$\langle \lambda_5^{(7)} \rangle$

(b) $n \neq 6$

G	$\mathrm{Comp}^{(n+1)}\, G$	G	$\mathrm{Comp}^{(n+1)}\, G$
$D_{[1,n-1]} \leq G$	$\langle \nu_{n-1}^{(n+1)} \rangle$	$D_{[2,n]} \leq G$	$\langle \lambda_{n-1}^{(n+1)} \rangle$
$D_{[1,n-2]} \leq G$	$\langle \nu_{n-2}^{(n+1)} \rangle$	$D_{[3,n]} \leq G$	$\langle \lambda_{n-2}^{(n+1)} \rangle$

(c) *Otherwise* $\mathrm{Comp}^{(n+1)}\, G \leq \langle \delta_{n+1} \rangle$.

(ii) $\mathrm{Comp}^{(n+2)}\, G \leq \langle \delta_{n+2} \rangle$.

The end

Thank you!