Relatively pseudocomplemented posets

Ivan Chajda – Helmut Länger

Palacký University Olomouc, Olomouc, Czech Republic,
email: ivan.chajda@upol.cz
TU Wien, Vienna, Austria,
email: helmut.laenger@tuwien.ac.at

Support of the research by the Austrian Science Fund (FWF), project I 1923-N25, and the Czech Science Foundation (GAČR), project 15-34697L, as well as by the project "Ordered structures for algebraic logic", supported by AKTION Austria – Czech Republic, project 75p11, is gratefully acknowledged.

92th Workshop on General Algebra, Prague, May 27, 2016

Contents:

1. Relative pseudocomplements

Relative pseudocomplements

Remark 1

$(B, \vee, \wedge, ', 0, 1)$ *Boolean algebra and* $a, b \in B \Rightarrow$
$\Rightarrow a' \vee b = \max\{x \in B \mid a \wedge x \leq b\}$

Definition 2

- $(S, \wedge)$: $\wedge$*-semilattice*
- $a, b \in S$
- *relative pseudocomplement* $a * b$ *of* a *with respect to* b $:=$
  $:= \max\{x \in S \mid a \wedge x \leq b\}$

Definition 3

- $(P, \leq)$: *poset*
- $a, b \in P$
- $L(a, b) := \{x \in P \mid x \leq a, b\}$, $L(a) := L(a, a)$

Relatively pseudocomplemented posets

Remark 4

$(S, \wedge)$ $\wedge$-*semilattice and* $a, x, b \in S \Rightarrow (a \wedge x \leq b \Leftrightarrow L(a, x) \subseteq L(b))$

Definition 5

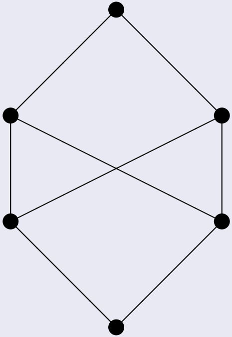
- $(P, \leq)$: *poset*
- $a, b \in P$
- *relative pseudocomplement* $a * b$ *of* a *with respect to* $b :=$
  $:= \max\{x \in P \mid L(a, x) \subseteq L(b)\}$
- $(P, \leq)$ *relatively pseudocomplemented* $:\Leftrightarrow \forall x, y \in P \exists x * y$
- $\mathcal{P} := \{$*relatively pseudocomplemented posets*$\}$

Remark 6

- *We write members of* $\mathcal{P}$ *in the form* $(P, \leq)$ *or* $(P, \leq, *)$.
- $(P, \leq) \in \mathcal{P} \not\Rightarrow (P, \leq)$ *semilattice*

Example of a member of $\mathcal{P}$ which is not a semilattice

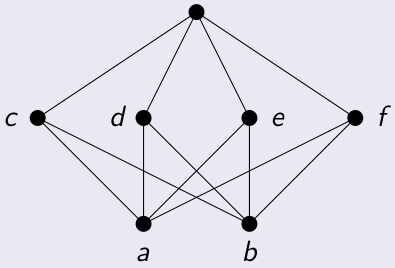
Example 7



is the Hasse diagram of a member of $\mathcal{P}$ *which is not a semilattice.*

Example of a poset not belonging to $\mathcal{P}$

Example 8



is the Hasse diagram of a poset not belonging to $\mathcal{P}$ *since*

$$\nexists c * d = \max\{x \mid L(c,x) \subseteq L(d)\} = \max\{a,b,d,e,f\}.$$

2. Properties of relatively pseudocomplemented posets

Properties of members of $\mathcal{P}$

Lemma 9

$(P, \leq) \in \mathcal{P}$ *and* $a, b, c \in P \Rightarrow$

- $(P, \leq)$ *has a greatest element* 1,
- $a \leq b \Leftrightarrow a * b = 1$,
- $a \leq b \Rightarrow (c * a \leq c * b$ *and* $b * c \leq a * c)$,
- $a \leq b * a$,
- $a \leq (a * b) * b$,
- $a \leq b * c \Leftrightarrow b \leq a * c$,
- $(a * b) * a \leq (a * b) * b$,
- $L(a, b) = L(a, a * b)$,
- $x * x \approx x * 1 \approx 1$,
- $1 * x \approx x$,
- $((x * y) * x) * y \approx ((x * y) * y) * y \approx x * y$.

Characterizing $\mathcal{P}$

Theorem 10

$(P, \leq)$ poset and $* : P^2 \to P \Leftrightarrow ((\mathrm{i}) \Leftrightarrow (\mathrm{ii}) \Leftrightarrow (\mathrm{iii}))$:

(i) $(P, \leq, *) \in \mathcal{P}$,

(ii) $(x \leq y * z \Leftrightarrow L(y, x) \subseteq L(z)) \forall x, y, z \in P$,

(iii) $((L(y, x) \subseteq L(z) \Rightarrow x \leq y * z)$ *and* $L(x, y) = L(x, x * y)) \forall x, y, z \in P$.

3. $\vee$-semilattices

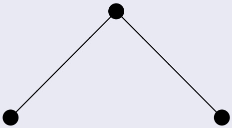
When is a member of $\mathcal{P}$ a $\vee$-semilattice?

Theorem 11

$(P, \leq, *) \in \mathcal{P}$

- $(x * y) * y \approx (y * x) * x \Rightarrow (P, \leq)$ *$\vee$-semilattice and* $x \vee y \approx (x * y) * y$
- $(P, \leq)$ *$\vee$-semilattice* $\not\Rightarrow (x * y) * y \approx (y * x) * x$

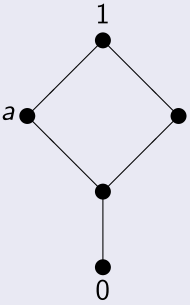
Example 12



is the Hasse diagram of a member of $\mathcal{P}$ *satisfying* $(x * y) * y \approx (y * x) * x$.

A ∨-semilattice belonging to $\mathcal{P}$ not satisfying $(x * y) * y \approx (y * x) * x$

Example 13



is the Hasse diagram of a ∨*-semilattice belonging to* $\mathcal{P}$*, but*

$$(0 * a) * a = 1 * a = a \neq 1 = 0 * 0 = (a * 0) * 0.$$

4. Distributive posets

Distributive posets

Definition 14

- $(P, \leq)$: *poset*
- $M, N \subseteq P$
- $L(M, N) := \{x \in P \mid x \leq y \forall y \in M \cup N\}$
- $U(M, N) := \{x \in P \mid x \geq y \forall y \in M \cup N\}$
- $(P, \leq)$ *distributive* $:\Leftrightarrow$
  $:\Leftrightarrow U(L(x, y), L(x, z)) = U(L(x, U(y, z))) \forall x, y, z \in P$

Lemma 15

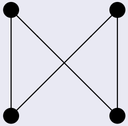
- $(L, \vee, \wedge)$ *lattice* $\Rightarrow ((L, \vee, \wedge)$ *distributive* $\Leftrightarrow (L, \leq)$ *distributive*)
- $(P, \leq)$ *poset* $\Rightarrow$ ((i) $\Leftrightarrow$ (ii)):
  (i) $(P, \leq)$ *distributive*
  (ii) $U(L(x, y), L(x, z)) \subseteq U(L(x, U(y, z))) \forall x, y, z \in P$

Distributive vs. relatively pseudocomplemented

Theorem 16

- $(P;\leq) \in \mathcal{P} \Rightarrow (P,\leq)$ *distributive*
- $(L,\vee,\wedge)$ *finite distributive lattice* $\Rightarrow (L,\leq) \in \mathcal{P}$
- $(P,\leq)$ *distributive* $\not\Rightarrow (P,\leq) \in \mathcal{P}$

Example 17



is the Hasse diagram of a distributive poset not belonging to $\mathcal{P}$ *since it has no greatest element.*

When do distributive posets belong to $\mathcal{P}$?

Theorem 18

$(P, \leq)$ *distributive and* $\mathrm{ACC} \Rightarrow (\mathrm{(i)} \Leftrightarrow \mathrm{(ii)})$*:*

(i) $(P, \leq) \in \mathcal{P}$

(ii) $\forall a, b \in P \exists$ *the supremum of any two maximal elements of* $\{x \in P \mid L(a, x) \subseteq L(b)\}$

5. Reference

Reference

[1] I. Chajda and H. Länger, Relatively pseudocomplemented posets. Math. Bohemica (submitted).

Thank you for your attention!