CSPs over the random partial order

Michael Kompatscher, Trung Van Pham

michael@logic.at

Institute of Computer Languages
TU Wien

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Outline

1. CSPs over the random partial order $\mathbb{P}$
2. Preclassification by homomorphic equivalence
3. Closed clones containing $\mathrm{Aut}(\mathbb{P})$
4. Results

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Poset-SAT

Φ... finite set of quantifier-free $\{\leq\}$-formulas

Poset-SAT(Φ)

Instance:

- Variables $\{x_1, \ldots, x_n\}$ and
- finitely many formulas $\phi_i(x_{i_1}, \ldots, x_{i_k})$, where each $\phi_i \in \Phi$.

Question:

Is $\bigwedge \phi_i(x_{i_1}, \ldots, x_{i_k})$ satisfiable in a partial order?

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For which Φ is Poset-SAT(Φ) in P?

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Poset-SAT($\perp$, Q)

$x \perp y :=$ incomparability relation
$Q(x, y, z) := (x < y \vee x < z)$

Poset-SAT($\perp$, Q) is NP-complete.

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$$R_\phi := \{(a_1, \ldots, a_n) \in P^n : \phi(a_1, \ldots, a_n)\}.$$

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$$\text{Poset-SAT}(\Phi) = \mathrm{CSP}((P; R_\phi)_{\phi \in \Phi}).$$

$(P; R_\phi)_{\phi \in \Phi}$ is a reduct of $\mathbb{P}$, i.e. a structure that is first-order definable in $\mathbb{P}$.

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Let $\mathrm{Pol}(\Gamma)$ denote the polymorphism clone of Γ, i.e. $f \in \mathrm{Pol}(\Gamma)$ if for all relations R of Γ: $\bar{r}_1, \ldots, \bar{r}_n \in R \to f(\bar{r}_1, \ldots, \bar{r}_n) \in R$.

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For ω-categorical structures Γ, Δ, every relation in Γ is pp-definable in Δ if

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$\rightarrow$ Aim: Understand the polymorphism clones of reducts of $\mathbb{P}$!

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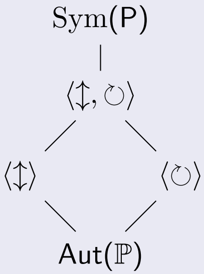
Automorphism groups

Theorem (Pach, Pinsker, Pongrácz, Szabó '14)

Let Γ be a reduct of $\mathbb{P}$. Then $\mathrm{Aut}(\Gamma)$ is equal to one of the following:



$\updownarrow$: bijection with $x < y \leftrightarrow \updownarrow x > \updownarrow y$

$\circlearrowright$: "rotation" at a generic upwards-closed set

Endomorphism monoids

Proposition (MK, Trung Van Pham '16)

Let Γ be reduct of $\mathbb{P}$. Then:

1. $\mathrm{End}(\Gamma)$ contains a constant,
2. $\mathrm{End}(\Gamma)$ contains $g_<$ that maps P to a chain $\cong \mathbb{Q}$,
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$\rightarrow$ We only need to study $\mathrm{CSP}(\Gamma)$, where $\overline{\mathrm{Aut}(\Gamma)} = \mathrm{End}(\Gamma)$.

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Polymorphisms of higher arity

Let $e_\leq : (P; \leq)^2 \to (P; \leq)$ be an embedding:

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By Bodirsky, Chen, Kára, von Oertzen '09

If $e_{\leq} \in \mathrm{Pol}(\Gamma)$ every relation in Γ has a $\leq$-Horn definition:

$$(x_{i_1} \leq x_{j_1}) \wedge (x_{i_2} \leq x_{j_2}) \cdots \wedge (x_{i_n} \leq x_{j_n}) \to (x_{i_{n+1}} \leq x_{j_{n+1}}) \text{ and}$$
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In this case $\mathrm{CSP}(\Gamma)$ is in P.

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Problem: How does $\mathrm{Pol}(\Gamma)$ look like? When is $e_\leq \in \mathrm{Pol}(\Gamma)$?

Canonical functions

A function $f : \Delta \to \Gamma$ is called canonical, if it maps tuples of the same *orbit* of $\mathrm{Aut}(\Delta) \curvearrowright \Delta^k$ to tuples of the same orbit of $\mathrm{Aut}(\Gamma) \curvearrowright \Gamma^k$.

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$\to$ Look for relations that imply NP-hardness.
$\to$ Use canonical functions for P.

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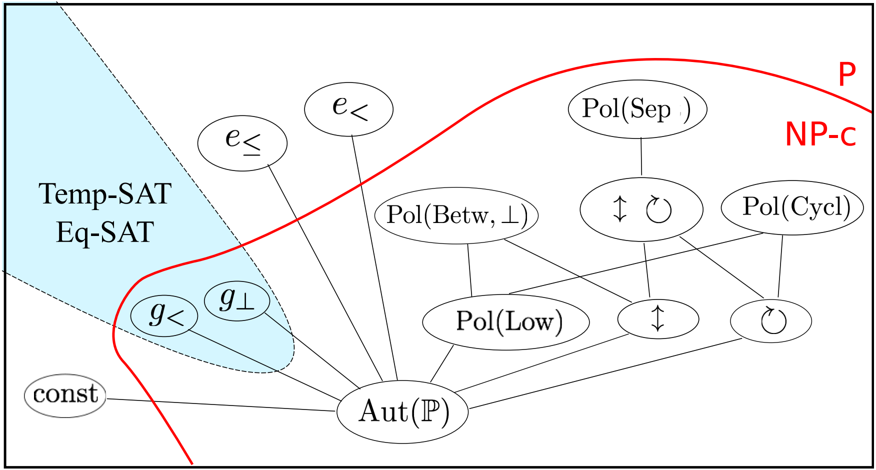
Consequence:

Poset-SAT(Φ) is in P or NP-complete.
Given Φ, it is decidable to tell if Poset-SAT(Φ) is in P.

Lattice of polymorphism clones



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  holds for $f \in \mathrm{Pol}(\Gamma), g_i \in \mathrm{End}(\Gamma)$ and $\mathrm{CSP}(\Gamma)$ is in P,

- or Γ is homomorphic equivalent to a Δ, such that:

$$\xi : \mathrm{Pol}(\Delta, c_1, \ldots, c_n) \to \mathbf{1}$$

  and $\mathrm{CSP}(\Gamma)$ is NP-complete.

Thank you!