Complemented quasiorder lattice of a monounary algebra

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Introduction

- **quasiorder of $\mathcal{A}$** = a binary relation on $\mathcal{A}$, which is
  - reflexive
  - transitive
  - compatible with all fundamental operations of $\mathcal{A}$

- **the lattice** $(\mathrm{Quord}(\mathcal{A}), \subseteq)$ **of all quasiorders of an algebra** $\mathcal{A}$

Introduction

- M. Erné and J. Reinhold (1995): lattices of all quasiorders on a **set**
  - atomistic
  - dually atomistic
  - complemented
- I. Chajda and G. Czédli (1996), A. G. Pinus (1995):
  - every algebraic lattice is isomorphic to the quasiorder lattice of a suitable algebra
- G. Czédli and A. Lenkehegyi (1983), A. G. Pinus and I. Chajda (1993):
  - quasiorder lattice of a majority algebra is always distributive
- R. Pöschel and S. Radeleczki:
  - how endomorphisms of quasiorders behave
  - when $\mathrm{End}\ q \subseteq \mathrm{End}\ q^{'}$ for quasiorders $q, q^{'}$ on a set A ($\mathrm{End}\ q$ is the set of all mappings preserving q)
  - description of the quasiorder lattice of the algebra $(A, \mathrm{End}\ q)$
- D. Jakubíková-Studenovská, R. Pöschel and S. Radeleczki:
  - irreducible quasiorders of monounary algebras

Introduction

- a monounary algebra $\mathcal{A} = (A, f)$ - can be depicted as a planar graph
- **an element** $x \in A$ is referred to as **cyclic** if there exists a positive integer n such that $f^n(x) = x$

AIM

- Construct a complementary quasiorder to a given quasiorder, if the lattice $\mathrm{Quord}(A, f)$ is complemented.

Necessary and sufficient condition

Theorem

Let (A, f) be a monounary algebra. The lattice $\mathrm{Quord}\,(A, f)$ is complemented if and only if

- *for each $a \in A$, the element $f(a)$ is cyclic,*
- *there is $n \in N$ such that each cycle of (A, f) has n elements,*
- *either $n = 1$ or n is square-free.*

Sufficiency of the condition was proved by means of transfinite induction. We will describe a **construction of a complement** to a given quasiorder of (A, f) satisfying this condition.

Preliminary

- **Assumption:** Let (A, f) be a monounary algebra such that
  - for each $a \in A$, the element $f(a)$ is cyclic,
  - there is $n \in N$ such that each cycle of (A, f) has n elements,
  - either $n = 1$ or n is square-free.
- Let $\alpha \in \operatorname{Quord}(A, f)$.
- For $\alpha \in \operatorname{Quord}(A, f)$, define $\bar{\alpha}$:

$$(b, a) \in \bar{\alpha} \Longleftrightarrow (a, b) \in \alpha.$$

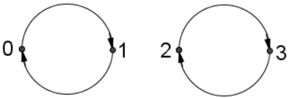
- For $a \in A$ denote by $C(a)$ the cycle, containing $f(a)$.

Preliminary

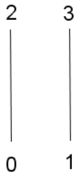
- Let r_α be the binary relation defined on the set of all cycles of (A, f) as follows: If B, D are cycles of (A, f), then we put $B \; r_\alpha \; D$, if there are $k \in \mathbb{N}$, cycles $B = C_0, C_1, \dots, C_k = D$, elements $c_0 \in C_0, c_1 \in C_1, \dots, c_k \in C_k$ such that for each $i \in \{0, 1, \dots, k-1\}$, $(c_i, c_{i+1}) \in \alpha \cup \bar{\alpha}$.
- If $a, b \in A$, then we set

$$a \; r_\alpha \; b \iff C(a) \; r_\alpha \; C(b).$$

- Let $A/r_\alpha = \{A_j : j \in J\}$. If J is a one-element set, then α is said to be **connected**.



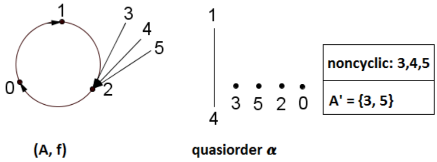
(A, f)



quasiorder α

Preliminary

- A': all noncyclic elements x of A such that $(x, f^n(x)) \notin \alpha \cup \bar{\alpha}$.



- ρ on A': $(a, b) \in \rho$ if $a, b \in A'$, $f(a) = f(b)$ and there are $k \in N$ and $a = u_0, u_1, \dots, u_k = b$ elements of A' such that $(\forall i \in \{0, \dots, k-1\})(f(a) = f(u_i), (u_i, u_{i+1}) \in \alpha \cup \bar{\alpha})$.
- ρ is an equivalence on A'.
- for each $D \in A'/\rho$ there are $D^* \subseteq D$ such that
  - $(\forall x \in D \setminus D^*)(\exists y \in D^*)((x, y) \in \alpha, (y, x) \in \alpha)$;
  - $(\forall x, y \in D^*, x \neq y)((x, y) \in \alpha \Rightarrow (y, x) \notin \alpha)$.
- We choose arbitrary D^* for each D and an arbitrary representative $d^* \in D^*$.

Construction of a complement to a connected quasiorder

- Let $\alpha \in \text{Quord}\,(A, f)$, be a connected quasiorder.
- Let A', ρ be as above.
- Let D^* and d^* be as fixed.
- Let $x, y \in A$. We put $(x, y) \in \beta$ if either $x = y$ or (x, y) fulfills one of the steps of the following Construction (K).

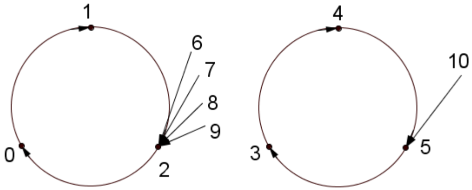
Construction (K)

- *Step (a).* Let x, y belong to the same cycle C, $y = f^k(x)$, $\alpha \restriction C = \theta_d$, d/n and let $e = \frac{n}{d}$. We set $(x, y) \in \beta$ if and only if e/k.
- *Step (b).* Let $x \in C_1$, $y \in C_2$, where C_1 and C_2 are distinct cycles. We put $(x, y) \in \beta$ if and only if there are $a \in C_1$ and $b \in C_2$ with $(b, a) \in \alpha, (a, b) \notin \alpha$.
- *Step (c).* Suppose that $x, y \in D^*$ for some $D \in A'/\rho$. Then $(x, y) \in \beta$ if and only if and $(y, x) \in \alpha$.
- *Step (d1).* Suppose that x belongs to a cycle C, y is noncyclic, $C(y) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$. If $y \notin A'$, then $(x, y) \in \beta$ if and only if $(f^n(y), y) \notin \alpha, (y, f^n(y)) \in \alpha, x = f^k(y), e/k$.

Construction (K)

- *Step (d'1).* Suppose that y belongs to a cycle C, x is noncyclic, $C(x) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$. If $x \notin A'$, then $(x, y) \in \beta$ if and only if $(f^n(x), x) \in \alpha, (x, f^n(x)) \notin \alpha, y = f^k(x), e/k.$
- *Step (d2).* Suppose that x belongs to a cycle C, y is noncyclic, $C(y) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$. If $y \in A'$, then $(x, y) \in \beta$ if and only if there is $D \in A'/\rho$ such that $y \in D^*, x = f^k(y), e/k$ and $(y, p(D)) \in \alpha$.
- *Step (d'2).* Suppose that y belongs to a cycle C, x is noncyclic, $C(x) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$. If $x \in A'$, then $(x, y) \in \beta$ if and only if there is $D \in A'/\rho$ such that $x \in D^*, y = f^k(x), e/k$ and $(x, p(D)) \in \alpha$.
- *Step (e).* Suppose that x, y satisfy none of the assumptions of the previous steps. Then $(x, y) \in \beta$ if and only if $(x, f^n(x)) \in \beta$, $(f^n(x), f^n(y)) \in \beta$, $(f^n(y), y) \in \beta$.

Let (A, f) be a given algebra:

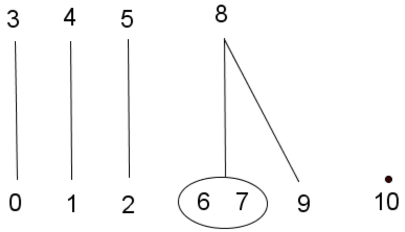


n is number of elements of each cycle.

- $n = 3$

Construction (K) - example

Let $\alpha \in \mathrm{Quord}(A, f)$ (connected):



A': all noncyclic elements x of A such that $(x, f^n(x)) \notin \alpha$ and $(f^n(x), x) \notin \alpha$.

- $A' = \{6, 7, 8, 9, 10\}$

Construction (K) - example

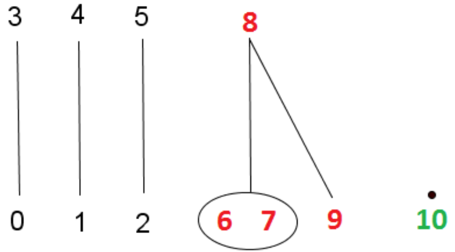
ρ on A': $(a,b) \in \rho$ if $a, b \in A'$, $f(a) = f(b)$ and a, b belong to the same connected subcomponent of the quasiordered set of α, consisting of elements of A'.

- ρ :

$6,7,8,9$	10

- A'/ρ :

D_1	$6,7,8,9$
D_2	10



Construction (K) - example

For each $D \in A'/\rho$ let us choose $D^* \subseteq D$ and $d^* \in D^*$ such that:

1) $(\forall x \in D \setminus D^*)(\exists y \in D^*)((x,y) \in \alpha, (y,x) \in \alpha)$;
2) $(\forall x, y \in D^*, x \neq y)((x,y) \in \alpha \Rightarrow (y,x) \notin \alpha)$.

A'/ρ :

D_1	$6, 7, 8, 9$
D_2	10

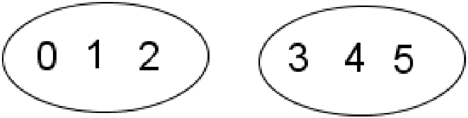
Let:

- $D_1^* = \{6, 8, 9\}$ and $d_1^* = 8$
- $D_2^* = \{10\}$ and $d_2^* = 10$

Construction (K) - example

Step (a). Let x, y belong to the same cycle C, $y = f^k(x)$, $\alpha \restriction C = \theta_d$, d/n and let $e = \frac{n}{d}$. We set $(x, y) \in \beta$ if and only if e/k.

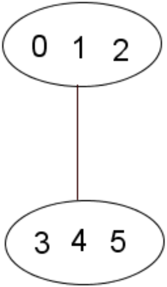
- It follows that $(x, y) \in \beta$ if and only if either $x, y \in \{0, 1, 2\}$, or $x, y \in \{3, 4, 5\}$.



Construction (K) - example

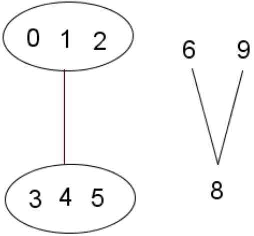
Step (b). Let $x \in C_1$, $y \in C_2$, where C_1 and C_2 are distinct cycles. We put $(x, y) \in \beta$ if and only if there are $a \in C_1$ and $b \in C_2$ with $(b, a) \in \alpha, (a, b) \notin \alpha$.

- It follows that $(x, y) \in \beta$ if and only if $x \in \{3, 4, 5\}$ and $y \in \{0, 1, 2\}$.



Step (c). Suppose that $x, y \in D^*$ for some $D \in A'/\rho$. Then $(x, y) \in \beta$ if and only if and $(y, x) \in \alpha$.

- We distinguish two cases:
  1. $x, y \in D_1^* = \{6, 8, 9\}$, then $(x, y) \in \beta$ if and only if $(x, y) \in \{(8, 6), (8, 9)\}$,
  2. $x, y \in D_2^* = \{10\}$, then $(x, y) \in \beta$ if and only if $(x, y) = (10, 10)$.



Step (d1).
Step (d'1).

- Both these steps operate with noncyclic elements $a \notin A'$, however, there are no such elements in (A, f).

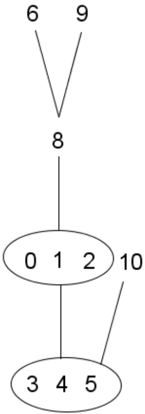
Construction (K) - example

Step (d2). Suppose that x belongs to a cycle C, y is noncyclic, $C(y) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$. If $y \in A'$, then $(x, y) \in \beta$ if and only if there is $D \in A'/\rho$ such that $y \in D^*, x = f^k(y), e/k$ and $(y, d^*) \in \alpha$.

- We distinguish two cases (for two cycles):
  1. $x \in \{0, 1, 2\}, y \in \{6, 7, 8, 9\}$.
  2. $x \in \{3, 4, 5\}, y = 10$.
- It follows that $(x, y) \in \beta$ if and only if $x \in \{0, 1, 2\}, y \in \{6, 8, 9\}$ or $x \in \{3, 4, 5\}, y = 10$.

Construction (K) - example

Step (d2).

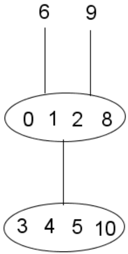


Construction (K) - example

Step (d'2). Suppose that y belongs to a cycle C, x is noncyclic, $C(x) = C$. Further let $\alpha \upharpoonright C = \theta_d$, d/n, $e = \frac{n}{d}$. If $x \in A'$, then $(x, y) \in \beta$ if and only if there is $D \in A'/\rho$ such that $x \in D^*, y = f^k(x), e/k$ and $(x, d^*) \in \alpha$.

- We distinguish two cases (for two cycles):
  1. $x \in \{6, 7, 8, 9\}, y \in \{0, 1, 2\}$.
  2. $x = 10, y \in \{3, 4, 5\}$.
- It follows that $(x, y) \in \beta$ if and only if $x = 8, y \in \{0, 1, 2\}$ or $x = 10, y \in \{3, 4, 5\}$.

Step (d'2).



Step (e). Suppose that x, y satisfy none of the assumptions of the previous steps. Then $(x, y) \in \beta$ if and only if $(x, f^n(x)) \in \beta$, $(f^n(x), f^n(y)) \in \beta$, $(f^n(y), y) \in \beta$.

- In this example, remaining cases are:
  1. x is a cyclic element, y is a noncyclic element from another cycle,
  2. x is a noncyclic element, y is a cyclic element from another cycle,
  3. x, y are noncyclic elements such that $x, y \notin D^*$ for any D^*.
- Then $(x, y) \in \beta$ if and only if $(x, f^3(x)) \in \beta$, $(f^3(x), f^3(y)) \in \beta$, $(f^3(y), y) \in \beta$.

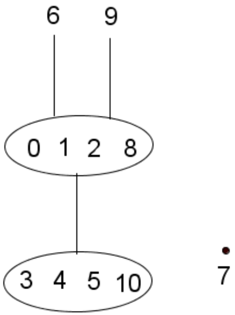
Construction (K) - example

Step (e). $(x, y) \in \beta$ if and only if $(x, f^3(x)) \in \beta$, $(f^3(x), f^3(y)) \in \beta$, $(f^3(y), y) \in \beta$. It follows that

1. If x is a cyclic element, y is a noncyclic element from another cycle, then $(x, y) \in \beta$ iff $x \in \{3, 4, 5\}$ and $y \in \{6, 8, 9\}$.
2. If x is a noncyclic element, y is a cyclic element from another cycle, then $(x, y) \in \beta$ iff $x = 10$ and $y \in \{0, 1, 2\}$.
3. x, y are noncyclic elements such that $x, y \notin D^*$ for any D^*, then $(x, y) \in \beta$ iff $x = 10$ and $y \in \{6, 8, 9\}$.

Construction (K) - example

We constructed a complementary quasiorder β to the quasiorder α.

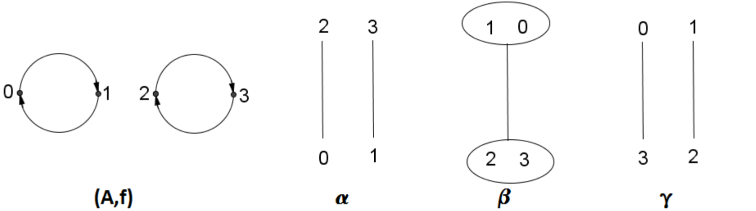


Connected quasiorders - main result

Theorem

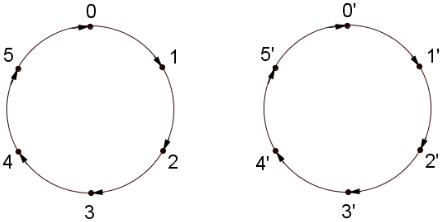
Let (A, f) be a monounary algebra whose lattice $\mathrm{Quord}\,(A, f)$ is complemented. Let $\alpha \in \mathrm{Quord}\,(A, f)$ be connected. If a binary relation β on A is formed by the Construction (K), then β is a complementary quasiorder to α in $\mathrm{Quord}\,(A, f)$.

The converse is not true:



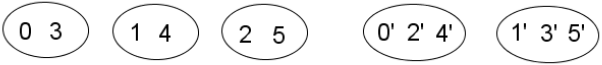
Construction of a complement - general case - example

- Let (A, f) be a given algebra:



Construction of a complement - general case - example

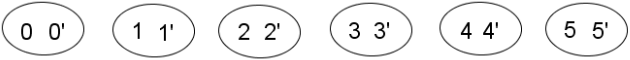
Let $\alpha \in \mathrm{Quord}\,(A, f)$, be a disconnected quasiorder, i.e let $A/r_\alpha = \{A_j : j \in J\}, |J| \geq 2$



- A/r_α :

A_1	$0, 1, 2, 3, 4, 5$
A_2	$0', 1', 2', 3', 4', 5'$

- For $i \in J$ let c_i be a fixed cyclic element of some chosen cycle C_i in A_i.
- Let $c_1 = 0, c_2 = 0'$.
- We define a relation $\gamma = \{(f^k(c_i), f^k(c_j) : i, j \in J, k \in \mathbb{N})\}$ (apparently a quasiorder):



Construction of a complement - general case - example

For each $i \in J$, the relation $\alpha \upharpoonright C_i$ is a congruence on C_i, thus $\alpha \upharpoonright C_i = \theta_{d_i}$.

- $\alpha_1 = \alpha \upharpoonright C_1 = \theta_3^1$, $d_1 = 3$
- $\alpha_2 = \alpha \upharpoonright C_2 = \theta_2^2$, $d_2 = 2$

$d =$gcd$(d_1, d_2) = 1$.

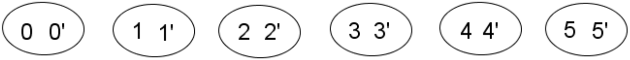
- $\alpha_1^{'} = \theta(c_1, f^d(c_1)) \vee \alpha_1 = \theta(0, 1) \vee \alpha_1 = \theta_1^1$
- $\alpha_2^{'} = \theta(c_2, f^d(c_2)) \vee \alpha_2 = \theta(0^{'}, 1^{'}) \vee \alpha_2 = \theta_1^2$

The quasiorder $\alpha_i^{'}$ is connected $\Rightarrow$ there exists $\beta_i^{'}$, a complementary quasiorder to α_i in $\mathrm{Quord}(A_i, f)$.

- $\beta_1^{'} = \Delta^1$
- $\beta_2^{'} = \Delta^2$

Construction of a complement - general case - example

- Let us define a relation
  $\beta = \gamma \ \vee \ \bigvee_{j \in J} \beta'_j = \gamma \vee (\Delta^1 \vee \Delta^2) = \gamma \vee \Delta = \gamma.$



- β is a complementary quasiorder to α in $\mathrm{Quord}\,(A, f)$.

Disconnected quasiorders - main result

Theorem

Let (A, f) *be a monounary algebra whose lattice* $\mathrm{Quord}\,(A, f)$ *is complemented. Let* $\alpha \in \mathrm{Quord}\,(A, f)$ *be disconnected. If a binary relation* β *on* A *is constructed as described, then* β *is a complementary quasiorder to* α *in* $\mathrm{Quord}\,(A, f)$.

Thank you for your attention.