Almost all strongly connected semicomplete digraphs are idempotent trivial

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Definitions

Definition

A digraph H is *semicomplete* if it is irreflexive (loopless) and for any two distinct vertices i and j, at least one of ij and ji is an edge of H. If $E(H)$ never contains both ij and ji, then it is a *tournament*.

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A semicomplete digraph $\mathcal{G} = (V, \to\}$ is strongly connected if for all $u, v \in G$ there exist $n \in \omega$ and vertices $a_1, \ldots, a_n \in V$ such that $u \to a_1 \to a_2 \to \ldots \to a_n \to v$.

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A *k-ary polymorphism* of a graph $\mathcal{H}$ is a homomorphism from $\mathcal{H}^k$ to $\mathcal{H}$. A polymorphism f is idempotent when, for all x, $f(x, \ldots, x) = x$.

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A tournament $\mathcal{T} = (T, \to)$ is *transitive* if from $x \to y$ and $y \to z$ follows $x \to z$.

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Definition

A tournament is *locally transitive* if for all $v \in T$, the induced subtournaments on the sets $v^- := \{x \in T : x \rightarrow v\}$ and $v^+ := \{x \in T : v \rightarrow x\}$ are transitive.

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A digraph is a *core* if all of its endomorphisms are automorphisms.

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For all finite core digraphs $\mathcal{G}$ with > 2 vertices which have no idempotent polymorphisms other than projections, QCSP$(\mathcal{G})$ is Pspace-complete.

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The goal of this lecture is

Theorem

If $\mathcal{G}$ is a strongly connected semicomplete digraph with more than one cycle, then QCSP$(\mathcal{G})$ is Pspace-complete.

Proof

Definition

A subset $L \subseteq V$ is nice if the induced subgraph on L is strongly connected and all idempotent polymorphisms of $\mathcal{G}$ restrict to L as projections.

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Lemma (A1)

Let L be a nice subset of V and let v be a vertex such that $v^+ \cap L \neq \emptyset \neq v^- \cap L$. Then $L \cup \{v\}$ is nice.

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Lemma (A2)

Let $L = \{a, b\}$ be compatible with (i. e. closed under) the idempotent polymorphisms of $\mathcal{G}$ and let $a \leftrightarrow b$. If $v \in V \setminus L$ is such that $v^+ \cap L \neq \emptyset \neq v^- \cap L$ then $\{a, b, v\}$ is nice.

Proof

Definition

Let $\mathcal{G} = (V, \to)$ be a strongly connected semicomplete digraph. We say that L *splits* $\mathcal{G}$ if $\emptyset \neq L \subsetneq V$ is a subset with the following properties:

1. $\{L, L^{\forall+}, L^{\forall-}\}$ is a partition of V and
2. for any 2-cycle $a \leftrightarrow b$ in $\mathcal{G}$, $\{a, b\}$ is contained in one of L, $L^{\forall+}$, or $L^{\forall-}$.

Proof

Lemma (A3)

Let $\mathcal{G} = (V, \rightarrow)$ *be a strongly connected semicomplete digraph which is not a cycle. Let* L_0 *be either a 2-cycle or a nice subset of* V*. Then either all idempotent polymorphisms of* $\mathcal{G}$ *are projections, or there exists a subset* $L \subseteq V$ *such that* L *splits* $\mathcal{G}$, $L_0 \subseteq L$ *and either the induced subgraph on* L *is a 2-cycle, or* L *is nice.*

Proof

Lemma (A3)

Let $\mathcal{G} = (V, \rightarrow)$ be a strongly connected semicomplete digraph which is not a cycle. Let L_0 be either a 2-cycle or a nice subset of V. Then either all idempotent polymorphisms of $\mathcal{G}$ are projections, or there exists a subset $L \subseteq V$ such that L splits $\mathcal{G}$, $L_0 \subseteq L$ and either the induced subgraph on L is a 2-cycle, or L is nice.

Lemma (A4,A5)

Let $\mathcal{G} = (V, \rightarrow)$ be a strongly connected semicomplete digraph which is not a P-graph and let L split $\mathcal{G}$. Then there exist vertices $a_0, a_1, b_0 \in V$ such that $a_1 \leftarrow a_0 \rightarrow b_0 \rightarrow a_1$ and that either

- A4 $b_0 \in L^{\forall -}$ *and* $a_0, a_1 \in L^{\forall +}$, *or*
- A5 $b_0 \in L^{\forall +}$ *and* $a_0, a_1 \in L^{\forall -}$.

Proof

Definition

A locally transitive tournament $\mathcal{T} = (\{1, \ldots, n\}, \rightarrow)$ is *regular* iff $n = 2k + 1$ for some positive integer k and for all $1 \leq i < j \leq 2k + 1$, $i \rightarrow j$ iff $j - i \leq k + 1$ (otherwise $j \rightarrow i$). In other words, in the unique (up to isomorphism) regular locally transitive tournament with $2k + 1$ vertices, $\varphi_{\mathcal{T}}(i) = i + k$ if $i \leq k + 1$, and $\varphi_{\mathcal{T}}(i) = i - k - 1$ if $i > k + 1$.

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Definition

The semicomplete digraph $\mathcal{G}_{\mathcal{T}} = (V, E)$ will be called a P-graph parametrized by the locally transitive tournament $\mathcal{T} = (\{1, \ldots, n\}, \to)$ if there exists a partition ρ of the vertex set V into nonempty subsets $A_1, \ldots, A_n$ such that for all $i \neq j$ and all $a \in A_i$ and $b \in A_j$, $ab \in E$ iff $i \to j$ in $\mathcal{T}$.

Proof

Lemma

Let $\mathcal{T} = (\{1, \ldots, n\}, \rightarrow)$ *be a locally transitive tournament. Then* $\rho := \ker \varphi_{\mathcal{T}}$ *is a congruence of* $\mathcal{T}$ *such that* $\mathcal{T}/\rho$ *is a regular locally transitive tournament* $\mathcal{T}'$, $\mathcal{T}$ *is a P-graph parametrized by* $\mathcal{T}'$, *and every P-graph parametrized by* $\mathcal{T}$ *is also a P-graph parametrized by* $\mathcal{T}'$.

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Lemma

Every idempotent polymorphism f *of a regular locally transitive tournament* $\mathcal{T} = (\{1, 2, \dots, 2k+1\}, \to)$, *where* $k > 1$, *is a projection.*

Proof

Theorem (B1)

Every idempotent polymorphism f of a P-graph $\mathcal{G}_\mathcal{T}$ parametrized by the locally transitive tournament $\mathcal{T}$ is a projection, except when $\mathcal{G}_\mathcal{T}$ is the 3-cycle.

Proof

Theorem (B1)

Every idempotent polymorphism f of a P-graph $\mathcal{G}_\mathcal{T}$ parametrized by the locally transitive tournament $\mathcal{T}$ is a projection, except when $\mathcal{G}_\mathcal{T}$ is the 3-cycle.

Lemma (B2)

Let $\mathcal{G} = (V, \rightarrow)$ be a strongly connected semicomplete digraph which contains at least one 2-cycle. Then for each 2-cycle $a \leftrightarrow b$ in $\mathcal{G}$, the set $\{a, b\}$ is closed with respect to all idempotent polymorphisms of $\mathcal{G}$ and each binary idempotent polymorphism of $\mathcal{G}$ restricted to $\{a, b\}$ is a projection.

Proof

Lemma (B3)

If a strongly connected tournament $\mathcal{G} = (V, \to)$ is not a P-graph and for all $v \in V$, all strong components of the induced subgraphs on v^+ and on v^- are of sizes 1 or 3, then there is a 3-cycle $a \to b \to c \to a$ in $\mathcal{G}$ such that all idempotent polymorphisms of $\mathcal{G}$ restrict to $\{a, b, c\}$ as projections.

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Lemma (B4)

A strongly connected semicomplete digraph with at most four vertices which is not a cycle has only trivial idempotent polymorphisms.

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Lemma (B4)

A strongly connected semicomplete digraph with at most four vertices which is not a cycle has only trivial idempotent polymorphisms.

Theorem

A strongly connected semicomplete digraph which is not a cycle has only trivial idempotent polymorphisms.

THANK YOU FOR YOUR ATTENTION!