Varieties of Rickart rings

Insa Cremer

University of Latvia

May 28, 2016

Outline

Motivation

- Rickart rings are an algebraic generalization of rings of bounded operators on a Hilbert space.

Motivation

- Rickart rings are an algebraic generalization of rings of bounded operators on a Hilbert space.
- Speed and Evans proved that commutative Rickart rings form a variety.

Motivation

- Rickart rings are an algebraic generalization of rings of bounded operators on a Hilbert space.
- Speed and Evans proved that commutative Rickart rings form a variety.

Can this result be generalized to a wider class of Rickart rings?

Motivation

- Rickart rings are an algebraic generalization of rings of bounded operators on a Hilbert space.
- Speed and Evans proved that commutative Rickart rings form a variety.

Can this result be generalized to a wider class of Rickart rings?

- Reduced Rickart rings

Motivation

- Rickart rings are an algebraic generalization of rings of bounded operators on a Hilbert space.
- Speed and Evans proved that commutative Rickart rings form a variety.

Can this result be generalized to a wider class of Rickart rings?

- Reduced Rickart rings
- Rickart rings

Motivation

- Rickart rings are an algebraic generalization of rings of bounded operators on a Hilbert space.
- Speed and Evans proved that commutative Rickart rings form a variety.

Can this result be generalized to a wider class of Rickart rings?

- Reduced Rickart rings
- Rickart rings
- One-sided Rickart rings

Rickart rings

Definition

A ring R is called Rickart ring if for every $a \in R$ there exist idempotents $e, f \in R$ such that

Rickart rings

Definition

A ring R is called Rickart ring if for every $a \in R$ there exist idempotents $e, f \in R$ such that

- $ax = 0 \iff ex = x$,

Rickart rings

Definition

A ring R is called Rickart ring if for every $a \in R$ there exist idempotents $e, f \in R$ such that

- $ax = 0 \iff ex = x$,
- $xa = 0 \iff xf = x$.

Focal operations

Definition ([2])

A unary operation $'$ on a ring R is called right focal operation if for every $a \in R$

Focal operations

Definition ([2])

A unary operation $'$ on a ring R is called right focal operation if for every $a \in R$

- a' is idempotent,

Focal operations

Definition ([2])

A unary operation $'$ on a ring R is called right focal operation if for every $a \in R$

- a' is idempotent,
- $ax = 0 \iff a'x = x$ for all $x \in R$,

Focal operations

Definition ([2])

A unary operation $'$ on a ring R is called right focal operation if for every $a \in R$

- a' is idempotent,
- $ax = 0 \iff a'x = x$ for all $x \in R$,
- $a'' = 1 - a'$.

Focal operations

Definition ([2])

A unary operation $'$ on a ring R is called right focal operation if for every $a \in R$

- a' is idempotent,
- $ax = 0 \iff a'x = x$ for all $x \in R$,
- $a'' = 1 - a'$.

Definition ([2])

A unary operation $`$ on a ring R is called left focal operation if for every $a \in R$

Focal operations

Definition ([2])

A unary operation $'$ on a ring R is called right focal operation if for every $a \in R$

- a' is idempotent,
- $ax = 0 \iff a'x = x$ for all $x \in R$,
- $a'' = 1 - a'$.

Definition ([2])

A unary operation $`$ on a ring R is called left focal operation if for every $a \in R$

- $a`$ is idempotent,

Focal operations

Definition ([2])

A unary operation $'$ on a ring R is called right focal operation if for every $a \in R$

- a' is idempotent,
- $ax = 0 \iff a'x = x$ for all $x \in R$,
- $a'' = 1 - a'$.

Definition ([2])

A unary operation $^{\backprime}$ on a ring R is called left focal operation if for every $a \in R$

- $a^{\backprime}$ is idempotent,
- $xa = 0 \iff xa^{\backprime} = x$ for all $x \in R$,

Focal operations

Definition ([2])

A unary operation $'$ on a ring R is called right focal operation if for every $a \in R$

- a' is idempotent,
- $ax = 0 \iff a'x = x$ for all $x \in R$,
- $a'' = 1 - a'$.

Definition ([2])

A unary operation $^{\backprime}$ on a ring R is called left focal operation if for every $a \in R$

- $a^{\backprime}$ is idempotent,
- $xa = 0 \iff xa^{\backprime} = x$ for all $x \in R$,
- $a^{\backprime\backprime} = 1 - a^{\backprime}$.

Focal Rickart rings

Proposition ([2])

A ring R is a Rickart ring if and only if it admits right and left focal operations.

Focal Rickart rings

Proposition ([2])

A ring R is a Rickart ring if and only if it admits right and left focal operations.

Definition

A focal (Rickart) ring is an algebra $\langle R, +, \cdot, -, ', `, 0, 1\rangle$, where

Focal Rickart rings

Proposition ([2])

A ring R is a Rickart ring if and only if it admits right and left focal operations.

Definition

A focal (Rickart) ring is an algebra $\langle R, +, \cdot, -, ', `, 0, 1\rangle$, where

- $\langle R, +, \cdot, -, 0, 1\rangle$ is a Rickart ring,

Focal Rickart rings

Proposition ([2])

A ring R is a Rickart ring if and only if it admits right and left focal operations.

Definition

A focal (Rickart) ring is an algebra $\langle R, +, \cdot, -, ', `, 0, 1\rangle$, where

- $\langle R, +, \cdot, -, 0, 1\rangle$ is a Rickart ring,
- $'$ is its right focal operation,

Focal Rickart rings

Proposition ([2])

A ring R is a Rickart ring if and only if it admits right and left focal operations.

Definition

A focal (Rickart) ring is an algebra $\langle R, +, \cdot, -, ', `, 0, 1\rangle$, where

- $\langle R, +, \cdot, -, 0, 1\rangle$ is a Rickart ring,
- $'$ is its right focal operation,
- $`$ is its left focal operation.

Commutative focal Rickart rings

Theorem (Speed, Evans)

A commutative ring $\langle R, +, \cdot, 0 \rangle$ *is a Rickart ring if and only if it admits a unary operation* $'$ *with*

Commutative focal Rickart rings

Theorem (Speed, Evans)

A commutative ring $\langle R, +, \cdot, 0\rangle$ is a Rickart ring if and only if it admits a unary operation $'$ with

- $aa' = 0$,

Commutative focal Rickart rings

Theorem (Speed, Evans)

A commutative ring $\langle R, +, \cdot, 0\rangle$ is a Rickart ring if and only if it admits a unary operation $'$ with

- $aa' = 0$,
- $aa'' = a$,

Commutative focal Rickart rings

Theorem (Speed, Evans)

A commutative ring $\langle R, +, \cdot, 0 \rangle$ *is a Rickart ring if and only if it admits a unary operation* $'$ *with*

- $aa' = 0$,
- $aa'' = a$,
- $(ab)' = a' + b' - a'b'$.

Commutative focal Rickart rings

Theorem (Speed, Evans)

A commutative ring $\langle R, +, \cdot, 0 \rangle$ is a Rickart ring if and only if it admits a unary operation $'$ with

- $aa' = 0$,
- $aa'' = a$,
- $(ab)' = a' + b' - a'b'$.

Corollary (Speed, Evans)

The class of commutative focal rings is a variety.

Reduced focal Rickart rings

Definition

A ring is called reduced if it has no non-zero nilpotent elements.

Reduced focal Rickart rings

Definition

A ring is called reduced if it has no non-zero nilpotent elements.

Proposition

In a reduced ring

$$ab = 0 \iff ba = 0.$$

Example: Associate rings

Definition

Let A be a subdirect sum of rings R_i without divisors of zero. The ring$A$ is called associate ring if for every $a \in A$ the element a^0 defined by

$$a_i^0 := \begin{cases} 0, & \text{if } a_i = 0 \\ 1, & \text{else} \end{cases}$$

belongs to A.

Example: Associate rings

Definition

Let A be a subdirect sum of rings R_i without divisors of zero. The ring$A$ is called associate ring if for every $a \in A$ the element a^0 defined by

$$a_i^0 := \begin{cases} 0, & \text{if } a_i = 0 \\ 1, & \text{else} \end{cases}$$

belongs to A.

Proposition

Every associate ring is a reduced Rickart ring with focal operation defined by

$$a' := 1 - a^0.$$

One-sided Rickart rings

Theorem

A unitary ring $\langle R, +, \cdot, 0, 1\rangle$ *is a right Rickart ring if and only if it admits a unary operation* $'$ *such that*

One-sided Rickart rings

Theorem

A unitary ring $\langle R, +, \cdot, 0, 1\rangle$ *is a right Rickart ring if and only if it admits a unary operation* $'$ *such that*

- $aa' = 0$,

One-sided Rickart rings

Theorem

A unitary ring $\langle R, +, \cdot, 0, 1\rangle$ *is a right Rickart ring if and only if it admits a unary operation* $'$ *such that*

- $aa' = 0$,
- $a'' = 1 - a'$,

One-sided Rickart rings

Theorem

A unitary ring $\langle R, +, \cdot, 0, 1 \rangle$ is a right Rickart ring if and only if it admits a unary operation $'$ such that

- $aa' = 0$,
- $a'' = 1 - a'$,
- $(a''b)'' = (a''b)''(ab)''$.

One-sided Rickart rings

Theorem

A unitary ring $\langle R, +, \cdot, 0, 1\rangle$ *is a right Rickart ring if and only if it admits a unary operation* $'$ *such that*

- $aa' = 0$,
- $a'' = 1 - a'$,
- $(a''b)'' = (a''b)''(ab)''$.

Corollary

The class of unitary right-focal Rickart rings is a variety.

One-sided Rickart rings

Theorem

A unitary ring $\langle R, +, \cdot, 0, 1\rangle$ *is a right Rickart ring if and only if it admits a unary operation* $'$ *such that*

- $aa' = 0$,
- $a'' = 1 - a'$,
- $(a''b)'' = (a''b)''(ab)''$.

Corollary

The class of unitary right-focal Rickart rings is a variety.

- The same holds for unitary left-focal rings.

Rickart rings

Theorem

A ring $\langle R, +, \cdot, 0\rangle$ is a Rickart ring if and only if it is unitary and it admits

Rickart rings

Theorem

A ring $\langle R, +, \cdot, 0\rangle$ *is a Rickart ring if and only if it is unitary and it admits*

- *a unary oparation* $'$ *with*

Rickart rings

Theorem

A ring $\langle R, +, \cdot, 0\rangle$ *is a Rickart ring if and only if it is unitary and it admits*

- *a unary oparation* $'$ *with*
  - $aa' = 0$,

Rickart rings

Theorem

A ring $\langle R, +, \cdot, 0\rangle$ is a Rickart ring if and only if it is unitary and it admits

- *a unary oparation $'$ with*
    - $aa' = 0$,
    - $a'' = 1 - a'$,

Rickart rings

Theorem

A ring $\langle R, +, \cdot, 0\rangle$ *is a Rickart ring if and only if it is unitary and it admits*

- *a unary operation* $'$ *with*
  - $aa' = 0$,
  - $a'' = 1 - a'$,
  - $(a''b)'' = (a''b)''(ab)''$,

Rickart rings

Theorem

A ring $\langle R, +, \cdot, 0 \rangle$ *is a Rickart ring if and only if it is unitary and it admits*

- *a unary operation* $'$ *with*
  - $aa' = 0$,
  - $a'' = 1 - a'$,
  - $(a''b)'' = (a''b)''(ab)''$,
- *a unary operation*$^{\backprime}$ *with*

Rickart rings

Theorem

A ring $\langle R, +, \cdot, 0\rangle$ is a Rickart ring if and only if it is unitary and it admits

- *a unary operation $'$ with*
  - $aa' = 0$,
  - $a'' = 1 - a'$,
  - $(a''b)'' = (a''b)''(ab)''$,
- *a unary operation$^{\backprime}$ with*
  - $a^{\backprime}a = 0$,

Rickart rings

Theorem

A ring $\langle R, +, \cdot, 0\rangle$ is a Rickart ring if and only if it is unitary and it admits

- *a unary oparation $'$ with*
  - $aa' = 0$,
  - $a'' = 1 - a'$,
  - $(a''b)'' = (a''b)''(ab)''$,
- *a unary oparation$`$ with*
  - $a`a = 0$,
  - $a`` = 1 - a`$,

Rickart rings

Theorem

A ring $\langle R, +, \cdot, 0\rangle$ *is a Rickart ring if and only if it is unitary and it admits*

- *a unary oparation* $'$ *with*
  - $aa' = 0$,
  - $a'' = 1 - a'$,
  - $(a''b)'' = (a''b)''(ab)''$,
- *a unary oparation*$^{\backprime}$ *with*
  - $a^{\backprime}a = 0$,
  - $a^{\backprime\backprime} = 1 - a^{\backprime}$,
  - $(ab^{\backprime\backprime})^{\backprime\backprime} = (ab)^{\backprime\backprime}(ab^{\backprime\backprime})^{\backprime\backprime}$.

Rickart rings

Theorem

A ring $\langle R, +, \cdot, 0\rangle$ is a Rickart ring if and only if it is unitary and it admits

- *a unary oparation $'$ with*
  - $aa' = 0$,
  - $a'' = 1 - a'$,
  - $(a''b)'' = (a''b)''(ab)''$,
- *a unary oparation$^{`}$ with*
  - $a^{`}a = 0$,
  - $a^{``} = 1 - a^{`}$,
  - $(ab^{``})^{``} = (ab)^{``}(ab^{``})^{``}$.

Corollary

The class of focal Rickart rings is a variety.

Reduced Rickart rings

Theorem

A right focal Rickart ring $\langle R, +, \cdot, -,', 0, 1\rangle$ *is reduced if and only if for all* $a \in R$

$$a'a = aa'.$$

Reduced Rickart rings

Theorem

A right focal Rickart ring $\langle R, +, \cdot, -, ', 0, 1 \rangle$ *is reduced if and only if for all* $a \in R$
$$a'a = aa'.$$

Corollary

The class of reduced focal Rickart rings is a variety.

Speed & Evans style axioms

Theorem

A ring $\langle R, +, \cdot, 0 \rangle$ is a reduced Rickart ring if and only if it is unitary and admits a unary operation $'$ with

Speed & Evans style axioms

Theorem

A ring $\langle R, +, \cdot, 0\rangle$ is a reduced Rickart ring if and only if it is unitary and admits a unary operation $'$ with

- $aa' = 0 = a'a$,

Speed & Evans style axioms

Theorem

A ring $\langle R, +, \cdot, 0\rangle$ is a reduced Rickart ring if and only if it is unitary and admits a unary operation $'$ with

- $aa' = 0 = a'a$,
- $a'' = 1 - a'$,

Speed & Evans style axioms

Theorem

A ring $\langle R, +, \cdot, 0\rangle$ *is a reduced Rickart ring if and only if it is unitary and admits a unary operation* $'$ *with*

- $aa' = 0 = a'a$,
- $a'' = 1 - a'$,
- $(ab)' = a' + b' - a'b'$.

Permutability and regularity

Definition

Let $\mathcal{A}$ be an algebra.

Permutability and regularity

Definition

Let $\mathcal{A}$ be an algebra.

- $\mathcal{A}$ is called congruence permutable if for all congruences θ, σ holds $\theta \circ \sigma = \sigma \circ \theta$.

Permutability and regularity

Definition

Let $\mathcal{A}$ be an algebra.

- $\mathcal{A}$ is called congruence permutable if for all congruences θ, σ holds $\theta \circ \sigma = \sigma \circ \theta$.
- $\mathcal{A}$ is called regular ir for all congruences θ, σ and for all $a \in \mathcal{A}$ from $[a]_\sigma = [a]_\theta$ follows $\sigma = \theta$.

Permutability and regularity

Definition

Let $\mathcal{A}$ be an algebra.

- $\mathcal{A}$ is called congruence permutable if for all congruences θ, σ holds $\theta \circ \sigma = \sigma \circ \theta$.
- $\mathcal{A}$ is called regular ir for all congruences θ, σ and for all $a \in \mathcal{A}$ from $[a]_\sigma = [a]_\theta$ follows $\sigma = \theta$.

Proposition

The varieties mentioned in this talk are permutable and regular.

Open questions

- Is every reduced Rickart ring isomorphic to an associate ring?

Open questions

- Is every reduced Rickart ring isomorphic to an associate ring?
- Which focal Rickart rings are subdirectly irreducible?

References

T.P. Speed, M.W. Evans: *A note on commutative Baer rings*, J. Aust. Math. Soc. 13 (1971), 1-6.

J. Cīrulis: *Extending the star order to Rickart rings*, Linear and Multilinear Algebra (2015)

Thank you for your attention

Questions?