ON THE LATTICE STRUCTURE OF FOULIS SEMIGROUPS

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1. FOULIS AND BAER SEMIGROUPS

Definition [D.J.Foulis (1960)]

A *Baer *-semigroup* is an algebra $(S, \cdot, 0, *, ')$, where

- $(S, \cdot, 0)$ is a semigroup with zero,
- $*$ is an involution (an idempotent semigroup antimorphism)
- every x' is a projection (self-adjoint idempotent),
- the principal ideal generated by x' is the right annihilator of the element x: $\quad x'y = y$ iff $xy = 0$.

The operation $'$ is necessarily unique.
The projections in its range are said to be *closed*;
the closed projections form an orthomodular lattice with orthocomplementation $'$ and ordering given by

$p \leq q$ iff $pq = p$ (iff $qp = p$).

A dual definition.

The "right annihilator" axiom: $x'y = y$ iff $xy = 0$.

The operation $'$ could equivalently be replaced by an operation $^{\backprime}$ where $x^{\backprime}$ is a projection and satisfies the "left annihilator axiom"

- the principal ideal generated by $x^{\backprime}$ is the left annihilator of x:

  $yx^{\backprime} = y$ iff $yx = 0$.

The closed projections are the same: $\mathrm{ran}^{\backprime} = \mathrm{ran}'$.

- In 1960, D.Foulis proved the 'coordinatization theorem': every OM lattice is isomorphic to the lattice of closed projections of some Baer *-semigroup.
- In 1972, T.S.Blyth and M.F.Janowitz suggested the term 'Foulis semigroup' for what Foulis called a Baer *-semigroup.
- In 1973, D.H.Adams shew that Foulis semigroups form a variety: the annihilator axiom is equivalent to the identity

  $x'y(xy)' = y(xy)'$.
- In 1978, M.P.Drazin introduced, on a proper involution semigroup, the so called *star order*:

  $a \preceq b$ iff $a^*a = a^*b$ and $aa^* = ba^*$.
- Lattice structure of Rickart *-rings [i.e., Foulis semigroups that happen to be an involution ring] under this order has been studied by M.F.Janowitz in 1983, and later by J.C̄irulis in 2015.
- Most of the results can be transferred to Foulis semigroups.

It turns out that presence of involution is not necessary for the coordinatization theorem: a class of ordinary Baer semigroups do the job.

Involution is not necessary also to characterize the star order on a Foulis semigroup.

Definition [M.F.Janowitz (1965)]

A *Baer semigroup* is an algebra $(S, \cdot, 0, ^{\backprime}, ^{\prime})$ such that

- $(S, \cdot, 0)$ is a semigroup with zero,
- $x^{\backprime}$ and $x^{\prime}$ are idempotents,
- the left and right annihilator axioms are fulfilled:

  $yx = 0$ iff $yx^{\backprime} = y$, $\quad xy = 0$ iff $x^{\prime}y = y$.

Both annihilator axioms may be replaced by the respective Adams identities

$$(yx)^{\backprime}yx^{\backprime} = (yx)^{\backprime}y, \quad x^{\prime}y(xy)^{\prime} = y(xy)^{\prime}.$$

Again, $1 := 0^{\backprime} = 0^{\prime}$ is the unity in S.
The ranges of the operations $^{\backprime}$ and $^{\prime}$ [now not unique!] still coincide.
Let P stand for the common range; call idemotents in P *closed*.

Definition

A Baer semigroup is said to be *strong* if, for all $p, q \in P$,

- $p^{\backprime} = p'$,
- $pq \in P$ iff $qp \in P$.

Example

The $(\cdot, 0, ^{\backprime}, ')$-subreduct of a Foulis semigroup is a strong Baer semigroup (with its closed projections in the role of closed idempotents).

Theorem

(a) In a strong Baer semigroup, the closed idempotents still form an orthomodular lattice with the ordering given by

$e \leq f$ iff $ef = e$ [iff $fe = e$].

(b) every OM lattice is isomorphic to the lattice of closed idempotents of a strong Baer semigroup.

2. STAR ORDER ON STRONG BAER SEMIGROUPS

Theorem

On a Foulis semigroup, a relation $\preceq$ is a star order if and only if

$$(*)\ a \preceq b \ \text{ iff } \ a^{\backprime\backprime} b = a = b a^{\prime\prime}.$$

Definition

The *star order* $\preceq$ on a strong Baer semigroup is defined by the condition $(*)$.

In the rest, let S be a strong Baer semigroup.

Star order: $a \preceq b$ iff $a^{\natural}b = a = ba''$.

Definition

The *left*, resp., *right* *star-order* on S is defined by

$$a \preceq_l b \quad \text{iff} \quad a^{\natural}b = a = b^{\natural}a,$$

resp.,

$$a \preceq_r b \quad \text{iff} \quad ba'' = a = ab''.$$

For all a, b, $a \preceq b$ iff $a \preceq_l b$ and $a \preceq_r b$.

[left/right Baer semigroups].

Proposition (for star order)

In S,

(a) 0 is the least element,

(b) $P = [0, 1]$,

(c) the order $\preceq$ agrees on P with the natural ordering of closed idempotents; in particular,

(d) meets and joins in P agree with those existing in S,

(e) S has the greatest element only if $S = P$.

The proposition holds true for left/right star-ordered strong Baer semigroups.

Theorem (for star order)

(a) Every initial segment of S is an orthomodular lattice, in which joins and meets agree with those existing on the whole S.

(b) Any segment $[0, x]$ of S is embedded into the sublattices $[0, x^{\backslash\backslash}]$ and $[0, x'']$ of P.

Again, the proposition holds true for left/right star-ordered strong Baer semigroups.
Even more, the segment $[0, x]_l$, resp., $[0, x]_r$, is isomorphic to $[0, x^{\backslash\backslash}]$, resp., $[0, x'']$.

Recall that $[0, x] = [0, x]_l \cap [0, x]_r$.

A subset of S is said to be *compatible* if any pair of its elements has an upper bound.
P is an example of a maximal compatible subset of S.

Theorem

Let M be a maximal compatible subset of S. Then
(a) M is a lattice isomorphic to an ideal of P,

(b) if M has a greatest element which is invertible, then the ideal coincides with P.

Likewise for one-sided star orders and one-side [opposite!] invertibility.

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