The second centralizer of a monounary algebra

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AIMS

- conditions for an operation f when the operation is uniquely determined by its second centralizer
- algorithm for describing f by means of the second centralizer

Preliminaries

Definition

For a nonempty set A, a mapping $f\colon A \to A$ is called a *unary operation* on A. The pair (A, f) is said to be a *monounary algebra*.

Definition

The *centralizer* of a monounary algebra (A, f) is the set $\mathcal{C}\,(A, f)$ of those mappings $g\colon A \to A$ which commute with the mapping f.

Definition

The *first centralizer* of (A, f): $\mathcal{C}_1(A, f) = \mathcal{C}(A, f)$.
The *second centralizer* of (A, f) is the set

$$\mathcal{C}_2(A, f) = \bigcap_{g \in \mathcal{C}_1(A, f)} \mathcal{C}_1(A, g).$$

Preliminaries

- f and g are *equivalent with respect to the first centralizer* if

$$f \approx_1 g \Longleftrightarrow C_1(A, f) = C_1(A, g)$$

- f and g are *equivalent with respect to the second centralizer* if

$$f \approx_2 g \Longleftrightarrow C_2(A, f) = C_2(A, g)$$

Theorem

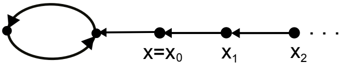
Mappings f and g of a set A into A are equivalent with respect to the first centralizer if and only if they are equivalent with respect to the second centralizer.

- The equality of the second centralizers do not imply the equality of the first centralizers of algebras (A, F) such that either F consists of an operation which is at least binary or $|F| > 1$.

Preliminaries

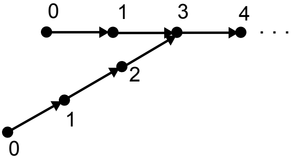
Definition

We will denote $s(x) = \infty$ if there exists a sequence $\{x_n\}_{n\in\mathbb{N}_0}$ of elements belonging to A with the property $x_0 = x$ and $f(x_n) = x_{n-1}$ for each $n \in \mathbb{N}$.



Definition

Next, $s(x) = k$, where $k \in \mathbb{N}_0$, if k is the largest element of $\mathbb{N}_0$ such that $f^{-k}(x) \neq \emptyset$.



Theorem 1

Let (A, f) be a connected monounary algebra. Then $C_2(A, f)$ uniquely determines f if and only if one of the following conditions is fulfilled:

(a) $s(x) \neq \infty$ for each $x \in A$,

(b) (A, f) contains a one-element cycle,

(c) (A, f) contains a k-element cycle with long tails, $k > 1$,

(d) (A, f) does not contain any cycle and there exist distinct elements $u, u', v, v' \in A$ such that $f(u) = u'$, $f(v) = v'$, $f(u') = f(v')$.

Theorem 2

Let (A, f) be a non-connected monounary algebra. Then $C_2(A, f)$ uniquely determines f if and only if one of the following conditions is fulfilled:

(a) A contains a component B such that $s(x) \neq \infty$ for each $x \in B$, or B does not contain any cycle and there exist distinct elements $u, u', v, v' \in B$ such that $f(u) = u'$, $f(v) = v'$, $f(u') = f(v')$,

(b) A contains a line with short tails and a k-element cycle with long tails, $k > 2$,

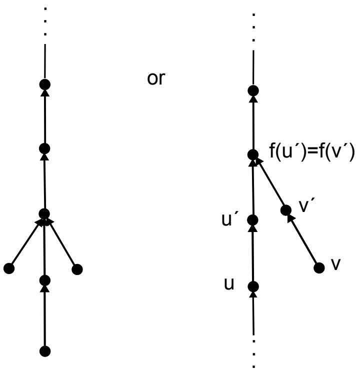
(c) A contains a line with short tails and a k-element cycle with an infinite tail, $k \in \{1, 2\}$,

(d) each component contains a cycle, where to each l-element cycle with short tails, $l > 2$, there is a k-element cycle with long tails, $k > 2$, such that

- either $l \mid k$,
- or $k \mid l$ and there is no $n \in \mathbb{N}$ with $1 < n < l$, $(n, l) = 1$, $n \equiv 1 (\mathrm{mod}\, k)$.

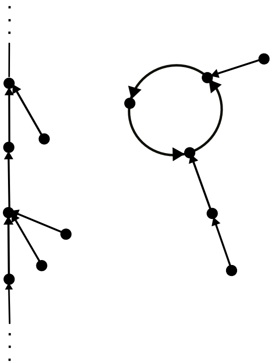
Theorem 2(a)

A contains a component B such that $s(x) \neq \infty$ for each $x \in B$, or B does not contain any cycle and there exist distinct elements $u, u', v, v' \in B$ such that $f(u) = u'$, $f(v) = v'$, $f(u') = f(v')$.



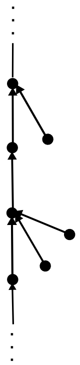
Theorem 2(b)

A contains a line with short tails and a k-element cycle with long tails, $k > 2$.



Theorem 2(c)

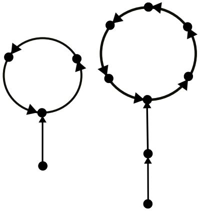
A contains a line with short tails and a k-element cycle with an infinite tail, $k \in \{1, 2\}$.



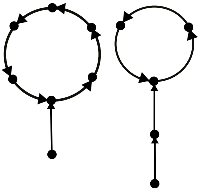
Theorem 2(d)

each component contains a cycle, where to each l-element cycle with short tails, $l > 2$, there is a k-element cycle with long tails, $k > 2$, such that

- either $l \,|\, k$,
- or $k \,|\, l$ and there is no $n \in \mathbb{N}$ with $1 < n < l$, $(n, l) = 1$, $n \equiv 1 (\mathrm{mod}\, k)$.



$l = 3$, $k = 6$



$l = 6$, $k = 3$

$n = 2$: $(2, 6) \neq 1$, $2 \not\equiv 1 \, (\mathrm{mod}\, 3)$
$n = 3$: $(3, 6) \neq 1$, $3 \not\equiv 1 \, (\mathrm{mod}\, 3)$
$n = 4$: $(4, 6) \neq 1$, $4 \equiv 1 \, (\mathrm{mod}\, 3)$
$n = 5$: $(5, 6) = 1$, $5 \not\equiv 1 \, (\mathrm{mod}\, 3)$

× 8-element cycle with short tails and 4-element cycle with long tails;
$l = 8$, $k = 4$; $n = 5$: $(5, 8) = 1$, $5 \equiv 1 \, (\mathrm{mod}\, 4)$

Notations

Let E be a monoid of transformations of a given set A.
Suppose that E is a centralizer of some monounary algebra defined on A.

- V_0 - a set of all cyclic elements of the algebra
- $V_1 = \{x \in A \setminus V_0 : f(x) \in V_0\}$
- Assume that we have defined V_i for each $i \leq n$, $n \in \mathbb{N}$.
- $V_{n+1} = \{x \in A : f(x) \in V_n\}$
- i-layer - V_i, $i \in \mathbb{N}$

- $V_0(C)$ - a cycle C from V_0.
- $V_0(c)$ - a one-element cycle (consisting of the element c)
- for $k \in \mathbb{N}$, the k-layer corresponding to C is the set $V_k(C) = \{x \in V_k : f(x) \in V_{k-1}(C)\}$
- the union of sets $V_i(C)$, $i \geq 0$ is the set of elements of the component containing the cycle C

Proposition

Let V_0^1 be a set of all one-element cycles in the algebra. Let $c \in A$.

(i) The element c belongs to V_0^1 if and only if there exists $\varphi \in E$ such that $\varphi(x) = c$ for all $x \in A$.

(ii) Let $x \in A$, $x \notin V_0^1$. Then $x \in V_1(c)$ and $f(x) = c$ if and only if there exists $\varphi \in E$ such that $\varphi^{-1}(c) = \{c, x\}$.

(iii) Let $x \notin V_0^1 \cup V_1^1$, $y \in V_1^1(c)$. Then $x \in V_2^1(c)$ and $f(x) = y$ if and only if there exists $\varphi \in E$ such that $\varphi^{-1}(c) = \{c, y\}$ and $\varphi(x) = y$.

(iv) Suppose that we have described operation f for all elements $z \in V_i^1(c)$, $i \leq n$. Let $x \notin V_0^1 \cup V_1^1 \cup \ldots \cup V_n^1$, $y \in V_n(c)$. Then $x \in V_{n+1}(c)$ and $f(x) = y$ if and only if there exists $\varphi \in E$ such that $\varphi^{-1}(V_1(c)) = \{y\}$ and $\varphi(x) \in V_2(c)$.

Notations

- Let V^1 be a set of all elements of components containing one-element cycle in the algebra.
- Let E_1 be a set of all $\varphi \in E$ such that $\varphi(x) \in A \setminus V^1$ for all $x \in A \setminus V^1$.
- A cycle B of length at least 2 is called a *minimal cycle* if $|D| = |B|$ or $|D| \nmid |B|$ is valid for every cycle D.
- A set $\{D_i : i \in I\}$ of cycles is said to be *minimal* if $|D_i| \neq |D_j|$ for $i \neq j$ and to every minimal cycle B there exists $k \in I$ such that $|B| = |D_k|$. In other words, the only representative is selected from the minimal cycles of the same length.
- If $D = \bigcup_{i \in I} D_i$, we say that D is a *minimal set of cyclic elements*.

Lemma

Let $\min_{\psi \in E_1} \left|\psi\left(A \setminus V^1\right)\right| = k$.

(a) C, $C \subseteq A$ is a minimal cycle, iff there exists $\varphi \in E_1$ such that $\left|\varphi\left(A \setminus V^1\right)\right| = k$, $C \subseteq \varphi\left(A \setminus V^1\right)$.

(b) The element $x \in A \setminus V^1$ belongs to some minimal cycle, iff there exists $\varphi \in E_1$ such that $\left|\varphi\left(A \setminus V^1\right)\right| = k$, $x \in \text{Im } \varphi$.

(c) The elements $x, y \in D$ belong to the same minimal cycle, iff there exists $\varphi \in E_1$ such that $\varphi(x) = y$.

(d) The elements $x, y \in A \setminus V^1$ belong to the same component iff for all $\varphi \in E_1$ if $\varphi(x) \neq x$ is from some minimal cycle then $\varphi(y) \neq y$.

(e) Let K be the component without a minimal cycle.
Consider $E_2 = \{\psi : \psi(z) = z, \forall z \notin K, \psi(y) \neq y, \forall y \in K\}$. The set of cyclic elements in K is $\bigcap_{\varphi \in E_2} \text{Im } (\varphi \restriction K)$.

Proposition

Let C be a cycle of length at least 2.

(i) There exists φ_0 such that $\varphi_0(c) \neq c$ for all $c \in C$ and the set $\{y : \varphi_0(y) \in C\}$ is minimal for all such mappings. Then $V_1(C) = \varphi_0^{-1}(C) \setminus C$.

(ii) $V_2(C) = \{t : \varphi_0(t) \in V_1(C)\}$.

(iii) Suppose that we have described i-layer $V_i(C)$ for $2 \leq i < n$. Then $V_n(C) = \{t : \varphi_0(t) \in V_{n-1}(C)\}$.

Proposition

Let C be a cycle of length at least 2 such that $V_2(C) = \emptyset$. Let $x \in V_1(C)$.

There exist $x' \in C$ and $\varphi \in E_1$ such that $\varphi(t) = \begin{cases} x' & \text{if } t \in \{x, x'\}, \\ t & \text{otherwise.} \end{cases}$

Then $f(x) = f(x')$ where f is one of the mappings $\varphi \in E_1$ which is a permutation consisting of a single one cycle.

Proposition

Let C be a cycle of length at least 2 such that $V_2(C) \neq \emptyset$.

(i) Let $z_2 \in V_2(C)$. There exists the only element $z_1 \in V_1(C)$ such that for all $\varphi \in E_1$ if $\varphi(z_1) \in C$ then $\varphi(z_2) \in V_0(C) \cup V_1(C)$. Hence $f(z_2) = z_1$.
There exists the only element $z_0 \in C$ such that for all $\varphi \in E_1$ if $\varphi(z_2) = z_1$ then $\varphi(z_1) = z_0$. Hence $f(z_1) = z_0$.
There exist $\varphi \in E_1$, $x' \in C$ such that $\varphi(z_1) = x'$, $\varphi(z_2) \in C$. Then $f(\varphi(z_2)) = x'$. Now we can describe operation on the cycle. Let $c \in C$, $\psi : \varphi(z_2) \longmapsto x'$; then $f(c) = \psi(c)$.

(ii) Suppose that we have already described $f(x)$ for $x \in V_i(C)$, $2 \leq i < n$. Let $z_n \in V_n(C)$. There exists the only element $z_{n-1} \in V_{n-1}(C)$ such that there exists $\psi \in E_1$ for which $\psi(z_{n-1}) = f^{n-2}(t) \in V_1(C)$; $\psi(y) \in C$; $\psi(z_n) \in V_2(C)$. Then $f(z_n) = z_{n-1}$.

Theorem

Let f be a mapping of A to A. Then f is uniquely determined by E if and only if f is constructed according to the above algorithms and f fulfills the condition (d) of Theorem 2.

Thank you for your attention.