Globals of graphs

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Wider context: Power constructions in universal algebra

Three main power constructions:

- Power algebras: every operation $f : A^n \to A$ can be extended to an n-ary operation f^+ on the powerset $\mathcal{P}(A)$
- Complex algebras: for any $n+1$-ary relation on A, an n-ary operation on the powerset $\mathcal{P}(A)$ can be defined
- Power relations: an n-ary relation on A can be extended to an n-ary relation on the powerset $\mathcal{P}(A)$

Power relations

Definition

if R is a binary relation on A then R^+ is a binary relation on $\mathcal{P}(A)$ such that

$$XR^+Y \iff (\forall x \in X)(\exists y \in Y)\ xRy \ \& \ (\forall y \in Y)(\exists x \in X)\ xRy.$$

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Definition

The *global* of a graph $G = (V, E)$, denoted by $\mathcal{P}(G)$, is the graph with the set of vertices $\mathcal{P}(V)$ (the powerset of V), whose edges are determined by: for all $X, Y \in \mathcal{P}(G)$
XE^+Y if and only if $(\forall x \in X)(\exists y \in Y)xEy$ and $(\forall y \in Y)(\exists x \in X)xEy$.

Global determinism of graphs

Definition

For a class K of graphs we say that it is *globally determined* if for all graphs G_1 and G_2 from K, $\mathcal{P}(G_1) \simeq \mathcal{P}(G_2)$ implies $G_1 \simeq G_2$.

- Drápal, Globals of unary algebras (1985)
- Baumann, Pöschel, Schmeichel, Power graphs (1994)

CCB graphs

- Finite graphs whose connected components are complete graphs (with loops) or complete bipartite graphs
- CCB graphs are only finite graphs whose graph algebras have finite equational basis

Globals of CCB graphs

Proposition

If G is a a CCB graph, then the global of G is a CCB graph.

Proposition

If $\mathcal{P}(G)$ is a a CCB graph, then G is a CCB graph.

Proposition

If G has n complete components and m bipartite components, then $\mathcal{P}(G)$ has 2^{n+m} complete components and $\frac{2^{n+2m}-2^{n+m}}{2}$ bipartite components.

Two useful theorems

Theorem (Goldblatt)

$\mathcal{P}(\sum_{i\in I} G_i) \simeq \prod_{i\in I} \mathcal{P}(G_i)$

Theorem (Lovász)

Let G_1, G_2 and H be graphs. If $G_1 \times H \simeq G_2 \times H$ and H has a loop, then $G_1 \simeq G_2$.

CCB graphs are globally determined

Theorem

The class of finite CCB graphs is globally determined

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- Induction on the number of components

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- Induction on the number of components
- $\mathcal{P}(H+G_1) \simeq \mathcal{P}(H+G_2) \Rightarrow \mathcal{P}(H) \times \mathcal{P}(G_1) \simeq \mathcal{P}(H) \times \mathcal{P}(G_2)$

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- $\mathcal{P}(H) \times \mathcal{P}(G_1) \simeq \mathcal{P}(H) \times \mathcal{P}(G_2) \Rightarrow \mathcal{P}(G_1) \simeq \mathcal{P}(G_2)$

Forests

- A tree is a connected graph without cycles
- A forest is a disjoint union of trees
- Trees (and forests) are bipartite graphs

Trees are globally determined

Proposition

Let $G = (V, E)$ be an undirected graph. If Y is adjacent to a leaf in $\mathcal{P}(G)$ and $y \in Y$, then $\{y\}$ is adjacent to a leaf in $\mathcal{P}(G)$).

Proposition

Let $G = (V, E)$ be a finite undirected connected graph and $Y = \{y_1, \ldots, y_r\}$, $r \geq 2$, be a neighbour of a leaf in $\mathcal{P}(G)$. Then

$$d(Y) > \max_{y_i \in Y} d(\{y_i\}).$$

Trees are globally determined

Proposition

Let $G = (V, E)$ be a finite tree and $u \in V$. If X is a neighbour of $\{u\}$ and $d(X) = 2^k - 1$ for $k \geq 2$, then X is a singleton. If $d(X) = 1$ then there is at least one singleton among the leaves of $\mathcal{P}(G)$ which are neighbours of $\{u\}$.

Theorem

The class of finite trees is globally determined.

Forests are globally determined

- If the global of a forest is given, it is possible to identify one component of that forest
- Therefore, the class of finite forests is globally determined
- As well as disjoint unions of tournaments, regular digraphs, bipartite tournaments with loops, …

Thank you for your attention!