Congruence lattices forcing nilpotency

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Guiding questions

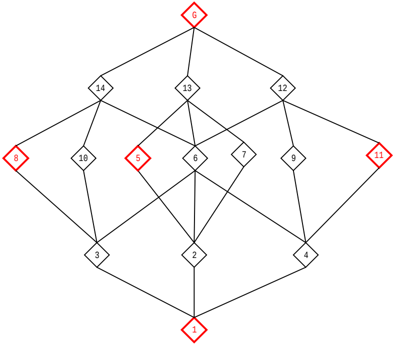
We are given the isomorphism class of the congruence lattice of an algebra. Must the algebra be

- abelian,
- supernilpotent,
- nilpotent,
- solvable?

Classic results: $\mathbb{M}_3$ as a sublattice

Theorem

Let **A** be an algebra in a cm variety. Assume that Con(**A**) has a $(0, 1)$-sublattice isomorphic to $\mathbb{M}_3$.
Then **A** is abelian, and hence polynomially equivalent to a module over a ring.



Classic results

Theorem

Let **A** be an algebra in a cm variety. Assume that Con(**A**) has a $(0, 1)$-sublattice $\mathbb{L}$ of finite height that is simple, complemented, and has at least 3 elements.
Then **A** is abelian.

Theorem (cf. [Hobby and McKenzie, 1988, Theorem 7.7])

Let **A** be a finite algebra in a cm variety. If Con(**A**) has no homomorphic image isomorphic to $\mathbb{B}_2$, then **A** is solvable.

Commutators

Theorem [Freese and McKenzie, 1987]

Let **A** be an algebra in a cm variety. For $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$, we define the commutator $[\alpha, \beta] := \ldots$.
Then $\mathbf{L} := (\mathrm{Con}(\mathbf{A}), \vee, \wedge, [.,.])$ satisfies

$$[x, y] \approx [y, x], \;\; [x, y] \leq x \wedge y, \;\; [\bigvee_{i \in I} x_i, y] \approx \bigvee_{i \in I} [x_i, y].$$

Definition of commutator lattices

Definition [Czelakowski, 2008]

$\mathbf{L} = (\mathbb{L}, \vee, \wedge, [.,.])$ is a *commutator lattice* if $(\mathbb{L}, \vee, \wedge)$ is a complete lattice, and $\mathbf{L}$ satisfies

$$[x, y] \approx [y, x], \;\; [x, y] \leq x \wedge y, \;\; [\bigvee_{i \in I} x_i, y] \approx \bigvee_{i \in I} [x_i, y].$$

Examples of $[.,.]$

- $\mathbb{L}$ complete lattice. $[x, y] := 0$ for all $x, y \in \mathbb{L}$.
- $\mathbb{L}$ finite distributive lattice. $[x, y] := x \wedge y$ for all $x, y \in \mathbb{L}$.

Properties of the commutator operation

A residuation operation

$(x : y) = \bigvee\{z \mid [z, y] \leq x\}$ for all $x, y \in \mathbb{L}$.

Think of $(x : y)$ as the *centralizer of y over x*.

Theorem (cf. [Czelakowski, 2008])

Let $\mathbf{L}$ be a commutator lattice. Then we have

$$(\bigwedge_{i \in I} x_i : y) \approx \bigwedge_{i \in I}(x_i : y), \quad (x : y) \geq x,$$
$$(x : \bigvee_{i \in I} y_i) \approx \bigwedge_{i \in I}(x : y_i), \quad (x : x) \approx 1, \quad (x : (x : y)) \geq y.$$

Properties of the commutator operation

Lemma

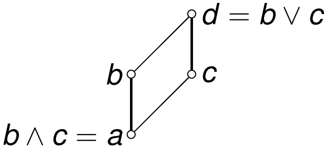
Let $\mathbb{L}$ be a commutator lattice. Let $a, b, c, d \in \mathbb{L}$ such that $a < b$, $c < d$, and $\mathbb{I}[a, b] \leftrightsquigarrow \mathbb{I}[c, d]$.
Then $(a : b) = (c : d)$ and $([b, b] \leq a \Leftrightarrow [d, d] \leq c)$.

Proof:

$(a : b) = (b \wedge c : b) = (b : b) \wedge (c : b) = (c : b) = (c : b) \wedge (c : c) = (c : b \vee c) = (c : d)$.

$\Rightarrow$: $[d, d] = [b \vee c, b \vee c] = [b, b] \vee [b, c] \vee [c, c] \leq a \vee c \vee c = c$.

$\Leftarrow$: $[b, b] \leq [d, d] \leq c$ and $[b, b] \leq b$, hence $[b, b] \leq b \wedge c = a$.



The largest commutator operation

Definition [Czelakowski, 2008]

Let $\mathbb{L}$ be a complete lattice. For $x, y \in \mathbb{L}$ we define

$$\lceil x, y \rceil := \bigvee_{j \in J} [x, y]_j,$$

where $\{[.,.]_j \mid j \in J\}$ is the set of all binary operations satisfying $[x, y] \approx [y, x]$, $[x, y] \leq x \wedge y$, and $[\bigvee_{i \in I} x_i, y] \approx \bigvee_{i \in I} [x_i, y]$.

Definition

Let $\mathbb{L}$ be a complete lattice. Let $\gamma_1 = \lambda_1 := 1$, $\gamma_{n+1} := \lceil \gamma_n, \gamma_n \rceil_{\mathbb{L}}$ and $\lambda_{n+1} := \lceil 1, \lambda_n \rceil$ for $n \in \mathbb{N}$.

- $\mathbb{L}$ *forces abelian type* if $\lceil 1, 1 \rceil = 0$.
- $\mathbb{L}$ *forces nilpotent type* if $\exists n \in \mathbb{N} : \lambda_n = 0$.
- $\mathbb{L}$ *forces solvable type* if $\exists n \in \mathbb{N} : \gamma_n = 0$.

The largest commutator operation - change of lattices

Theorem ($\mathbb{L} \leq_{0,1} \mathbb{K}$)

Let $\mathbb{L}$ be a complete lattice, and let $\mathbb{K}$ be a complete lattice such that $\mathbb{L}$ is a complete $(0, 1)$-sublattice of $\mathbb{K}$.
If $\mathbb{L}$ forces abelian, nilpotent, or solvable type, then so does $\mathbb{K}$.

Theorem ($\mathbb{L} \twoheadrightarrow \mathbb{K}$)

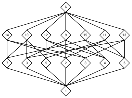
Let $\mathbb{L}$ be a complete lattice, and let $\mathbb{K}$ be a complete $(0, 1)$-homomorphic image of $\mathbb{L}$.
If $\mathbb{L}$ forces abelian, nilpotent, or solvable type, then so does $\mathbb{K}$.

Lattices forcing various types

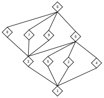
Theorem

Let $\mathbb{L}$ be a modular lattice of finite height. Then

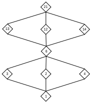
- $\mathbb{L}$ forces abelian type $\Leftarrow$ $\mathbb{L}$ has a $(0,1)$-sublattice with more than 2 elements that is simple and complemented.
- $\mathbb{L}$ forces solvable type $\Leftrightarrow$ $\mathbb{B}_2$ is not a homomorphic image of $\mathbb{L}$.
- $\mathbb{L}$ forces nilpotent type $\Leftarrow$ for all $\alpha \prec \beta \in \mathbb{L}$:
  $\bigvee\{\eta \in \mathbb{L} \,|\, \eta$ is meet irreducible and $\mathbb{I}[\alpha,\beta] \leftrightsquigarrow \mathbb{I}[\eta,\eta^{+}]\} = 1$.



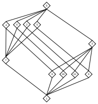
abelian type



nilpotent type



solvable type



Solvable algebras

Theorem

Let **A** be an algebra in a cm variety. If Con(**A**) has a complete $(0, 1)$-sublattice $\mathbb{L}$ of finite height such that $\mathbb{B}_2$ is not a homomorphic image of $\mathbb{L}$, then **A** is solvable.

Lemma

Let **A** be such that Con(**A**) is a modular lattice of finite height with $\mathbb{B}_2$ as a homomorphic image. Then there is $[.,.]$ such that $(\text{Con}(\mathbf{A}), \vee, \wedge, [.,.])$ is a commutator lattice with $\alpha \in \text{Con}(\mathbf{A})$ such that $\alpha \neq 0$ and $[\alpha, \alpha] = \alpha$.

Question

Can we realize this $[.,.]$ as the commutator operation of an expansion of **A**?

Solvable algebras

Theorem

Let $\mathbf{A}$ be an algebra in a cm variety such that $\mathrm{Con}(\mathbf{A})$ is of finite height and has $\mathbb{B}_2$ as a homomorphic image. Let

$$\mathbf{A}^c := (A, \mathrm{Pol}(\mathrm{Con}(\mathbf{A}))).$$

Then $\mathrm{Con}(\mathbf{A}) = \mathrm{Con}(\mathbf{A}^c)$ and $\mathbf{A}^c$ is not solvable.

Nilpotent algebras

Definition

Let $\mathbb{L}$ be a complete lattice. For $\alpha \prec \beta \in \mathbb{L}$, we define

$$\gamma(\alpha, \beta) := \bigvee \{\eta \in \mathbb{L} \,|\, \eta \text{ is m.i. and } \mathbb{I}[\eta, \eta^+] \leftrightsquigarrow \mathbb{I}[\alpha, \beta]\}.$$

Theorem

Let $\mathbf{A}$ be an algebra in a cm variety. Assume that Con($\mathbf{A}$) has a complete $(0, 1)$-sublattice $\mathbb{L}$ of finite height such that for all $\alpha, \beta \in \mathbb{L}$ with $\alpha \prec \beta$, we have $\gamma(\alpha, \beta) = 1$.
Then $\mathbf{A}$ is nilpotent.

Nilpotent algebras

Theorem

Let $\mathbf{A}$ be a finite expanded group, and let $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$ be such that $\alpha \prec \beta$. Then the centralizer $(\alpha : \beta)_{\mathbf{A}^c}$ of β over α in $\mathbf{A}^c$ is $\gamma(\alpha, \beta)$.

Corollary

Let $\mathbf{A}$ be a finite expanded group. Then $\mathbf{A}^c$ is nilpotent $\Leftrightarrow$ for all $\alpha, \beta \in \mathrm{Con}(\mathbf{A}) : (\alpha \prec \beta \Rightarrow \gamma(\alpha, \beta) = 1_A)$.

Missing Theorems

Missing Theorem: lattice side, abelian

Let $\mathbb{L}$ be a finite lattice. Then $\mathbb{L}$ forces abelian type if and only if $\mathbb{L}$ satisfies `Missing Condition 1`.

Missing Theorem: algebra side, abelian

Let $\mathbf{A}$ be a finite algebra in a cp variety. Then $\mathbf{A}^c$ is abelian if and only if Con($\mathbf{A}$) satisfies `Missing Condition 2 (or 1)`.

Missing Theorems

Conjecture: lattice side, nilpotent

Let $\mathbb{L}$ be a finite modular lattice. Then $\mathbb{L}$ forces nilpotent type $\Leftrightarrow$ $\gamma(\alpha, \beta) = 1$ for all $\alpha \prec \beta$.

Remarks: $\Leftarrow$ is proved. $\Rightarrow$ is true if $\mathbb{L}$ is the congruence lattice of a finite expanded group **A**. Then the commutator operation $[.,.]_{\mathbf{A}^c}$ of $\mathbf{A}^c$ is a lower bound for $\lceil .,. \rceil_{\mathbb{L}}$.

Conjecture: algebra side, nilpotent

Let **A** be an algebra in a cm variety with Con(**A**) of finite height. Then $\mathbf{A}^c$ is nilpotent $\Leftrightarrow \gamma(\alpha, \beta) = 1$ for all $\alpha \prec \beta \in$ Con(**A**).

$\Leftarrow$ is proved. $\Rightarrow$ is true if **A** is a finite expanded group.

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